WIRELESS COMMUNICATION TECHNOLOGIES

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CONTINUOUS PHASE MODULATION (CPM)

In order to further improve bandwidth efficiency, some of the mobile radio systems use special kind of frequency modulation schemes called continuous phase modulation - CPM. The schemes are particularly attractive because they have constant envelope and excellent spectral characteristics resulting from phase changes in a continuous manner.

The complex envelope of any CPM signal can be represented as:

$$v(t) = Ae^{j2\pi k_f \int_{-\infty}^{t} \sum_n x_n h_f(\tau - nT)d\tau} = Ae^{j\Phi(t)} \tag{1}$$

where,

A is the amplitude,
k_f is the peak frequency deviation,
$h_f(t)$ is the frequency shaping pulse,
$\{x_n\}$ where $x_n \in \{\pm 1, \pm 3, ..., \pm (M-1)\}$ is the source symbol sequence,
M is the symbol alphabet size, and
T is the symbol duration.

The phase term $\Phi(t)$ for the time interval $kT \le t \le (k+1)T$ can be rewritten as,

$$\Phi(t) = 2\pi k_f \int_{-\infty}^{kT} \sum_{n=-\infty}^{K-1} x_n h_f(\tau - nT)d\tau + 2\pi k_f \int_{kT}^{t} x_k h_f(\tau - kT)d\tau \tag{2}$$

where the first term corresponds to *accumulated phase* and the second term to *current phase* from the beginning of current symbol period up to time *t*. Sometimes, $\Phi(t)$ is called *excess phase*. The phase is therefore continuous as long as the frequency shaping function $h_f(t)$ does not contain impulses.

The frequency shaping function can be arbitrarily chosen. However, for the purpose of our analysis, we will divide it in two possible forms:

- **full response,** when the duration of $h_f(t)$ is equal to symbol period T, and
- **partial response**, when the duration of $h_f(t)$ is greater than symbol period T, thus is extended across several symbols.

The equation (2) represents the excess phase for the time interval $kT \leq t \leq (k+1)T$ and combining all such intervals we can write $v(t)$ in the standard form:

$$v(t) = A\sum_k b(t - kT, x_k) \tag{3}$$

where

$$b(t, x_k) = e^{j(\beta(T)\sum_{n=-\infty}^{k-1} x_n + x_k \beta(t))} u_T(t) \tag{4}$$

and

$$\beta(t) = \begin{cases} 0 & ,t < 0 \\ 2\pi k_f \int_0^t h_f(\tau)d\tau & ,0 \leq t \leq T \\ \beta(T) & ,T < t \end{cases} \tag{5}$$

The first term in the exponent of equation (4) represents the accumulated excess phase, while the second term is the excess phase trajectory for the current source symbol.

Two parameters characterize this type of modulation:

- **Average frequency deviation:**

$$\bar{k}_f = \frac{k_f}{T}\int_0^T h_f(t)dt \tag{6}$$

- **Modulation index:**

$$h = \frac{\beta(T)}{\pi} = 2k_f T \frac{1}{T}\int_0^T h_f(t)dt = 2\bar{k}_f T \tag{7}$$

By choosing different shaping pulses *$h_f t)$*, modulation indices *h* and alphabet sizes *M* we can generate an infinite variety of different classes of CPM signals. Here, we will describe four important schemes:

- Continuous phase frequency shift keying (CPFSK)
- Minimum shift keying (MSK)
- Partial response CPM
- Gaussian minimum shift keying (GMSK)

Continuous Phase Frequency Shift Keying (CPFSK)

For CPFSK,

$h_f(t) = u_T(t)$ – frequency shaping function is rectangular function,

$\bar{k}_f = k_f$ – average frequency deviation is equal to peak frequency deviation,

$h = 2k_f T$, and

$$\beta(t) = \begin{cases} 0 & ,t < 0 \\ 2\pi k_f t = \pi h t / T & ,0 \le t \le T \\ 2\pi k_f T = \pi h & ,T < t \end{cases} \tag{8}$$

Since $\beta(t)$ is continuous function of time, the CPM signals cannot be represented as discrete points in the signal space diagram. Therefore, the CPM signals are represented by sketching the excess phase (9) for all possible symbol sequences $\{x_k\}$. The plot is called a **phase tree**.

$$\Phi(t) = \beta(T) \sum_{n=-\infty}^{k-1} x_n + x_k \beta(t - kt) \tag{9}$$

Note that since the shaping function is rectangular, the phase changes are linear. For example, the binary phase tree is shown in Figure 1.

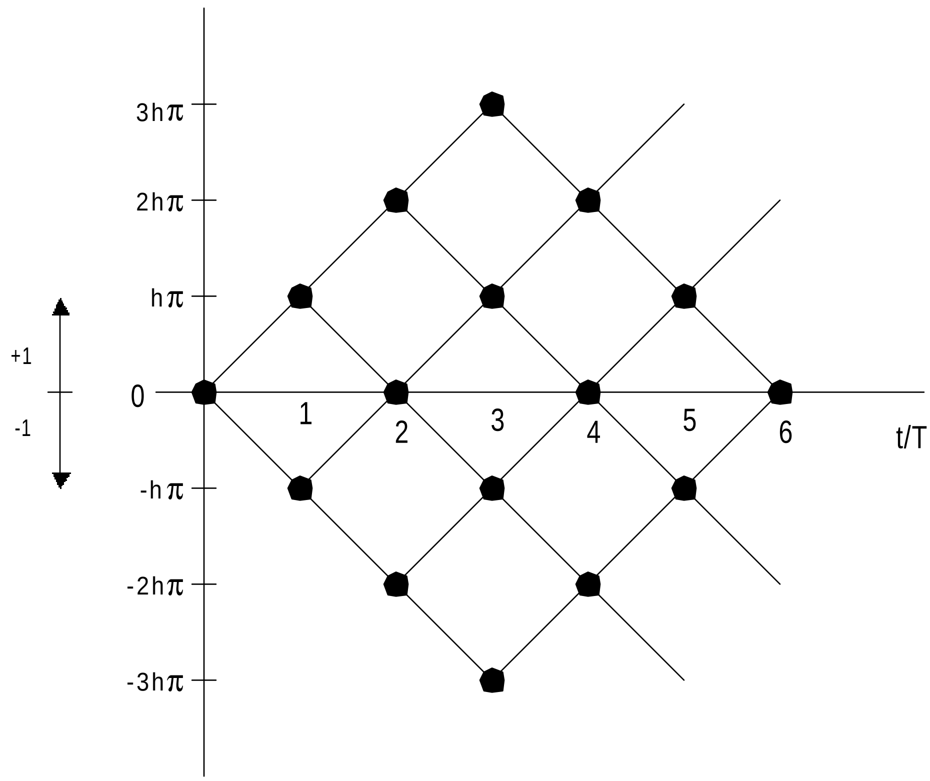

Figure 1: Phase tree of binary CPFSK with an arbitrary modulation index

Minimum Shift Keying (MSK)

MSK is a special case of binary CPFSK, with modulation index *h*=0.5. For MSK

$$\beta(t)=\begin{cases}0 & ,t<0\\ 2\pi k_f t=\pi t/2T & ,0\le t\le T\\ 2\pi k_f T=\pi/2 & ,T<t\end{cases} \tag{10}$$

The carrier phase, $\Phi_i(t) = 2\pi f_c t + \Phi(t)$, during interval $kT \le t \le (k+1)T$ is

$$\Phi_i(t)=2\pi f_c t+\frac{\pi}{2}\sum_{n=-\infty}^{k-1}x_n+\frac{\pi}{2}x_k\frac{t-kT}{T} \tag{11}$$

Rearranging,

$$\Phi_i(t) = (2\pi f_c + \frac{\pi x_k}{2T})t + \frac{\pi}{2}\sum_{n=-\infty}^{k-1} x_n - \frac{\pi k}{2} x_k \qquad (12)$$

Now, during time interval *kT ≤ t ≤ (k+1)T*, the MSK band-pass signal can be represented as,

$$s(t) = A\cos[(2\pi f_c + \frac{\pi x_k}{2T})t + \frac{\pi}{2}\sum_{n=-\infty}^{k-1} x_n - \frac{\pi k}{2} x_k] =$$

$$s(t) = A\cos[2\pi(f_c + \frac{x_k}{4T})t + \frac{\pi}{2}\sum_{n=-\infty}^{k-1} x_n - \frac{\pi k}{2} x_k] \qquad (13)$$

From the above, we can see that in time interval *kT ≤ t ≤ (k+1)T* MSK signal has one of two possible frequencies,

$$f_L = f_c - \frac{1}{4T} \quad \text{and} \quad f_U = f_c + \frac{1}{4T}$$

The difference between these frequencies is $\Delta f = f_U - f_L = \frac{1}{2T}$. This is the minimum frequency separation required to insure orthogonality with coherent modulation, hence the name *minimum* shift keying. The power spectral density of MSK signal is depicted in Figure 2.

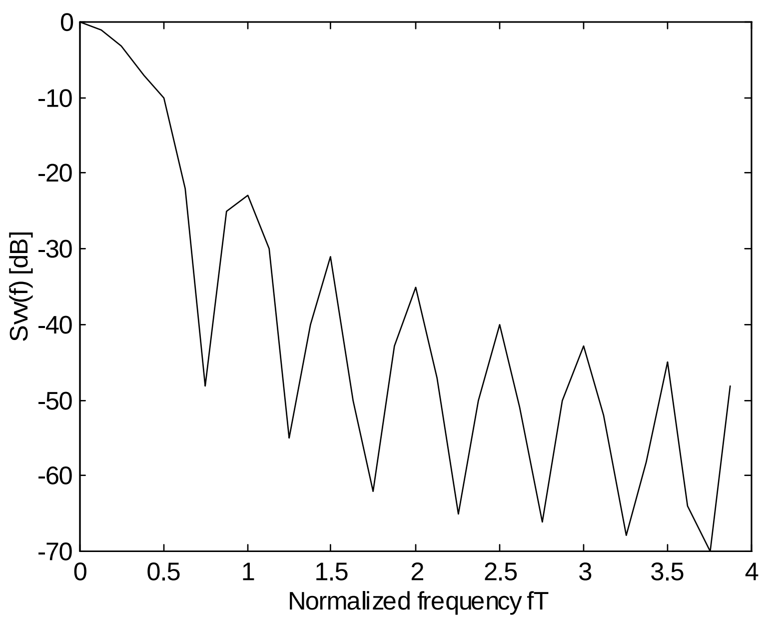

Figure 2: Power density spectra of MSK

Partial Response CPM

Partial response CPM signals are a broad class of signals characterized by a frequency shaping pulse $h_f(t)$ of duration greater that symbol period T. If $h_f(t)$ has duration KT, then

$$h_f(t) = h_f(t) u_{KT}(t) \tag{14}$$

where

$$u_{KT}(t) = \sum_{k=0}^{K-1} u_T(t - kT) \tag{15}$$

The advantage of using partial response CPM is an improvement in the spectral characteristics of the modulated signal by providing both a narrower main lobe and faster roll-off of side lobes. Combining (14) and (15), the frequency shaping function can be written as:

$$h_f(t) = \sum_{k=0}^{K-1} h_f(t) u_T(t - kT) \tag{16}$$

Gaussian Minimum Shift Keying (GMSK)

GMSK is a special case of partial response CPM that uses a low-pass pre-modulation filter with transfer function:

$$H(f) = e^{-\left(\frac{f}{B}\right)^2 \frac{\ln 2}{2}} \tag{17}$$

where B is the bandwidth of the filter. Since $H(f)$ is the bell-shaped about $f=0$, the modulation scheme has been named *Gaussian MSK*. The frequency shaping pulse is usually described with BT – normalized filter bandwidth. Note that when BT decreases, the pulse has duration greater than one symbol period T, thus introducing ISI. For this reason, systems using GMSK (like GSM) must have strong equalizer. The relation between MSK and GMSK is illustrated using power spectral densities with GMSK BT product equal BT=0.3 sketched in Figure 3.

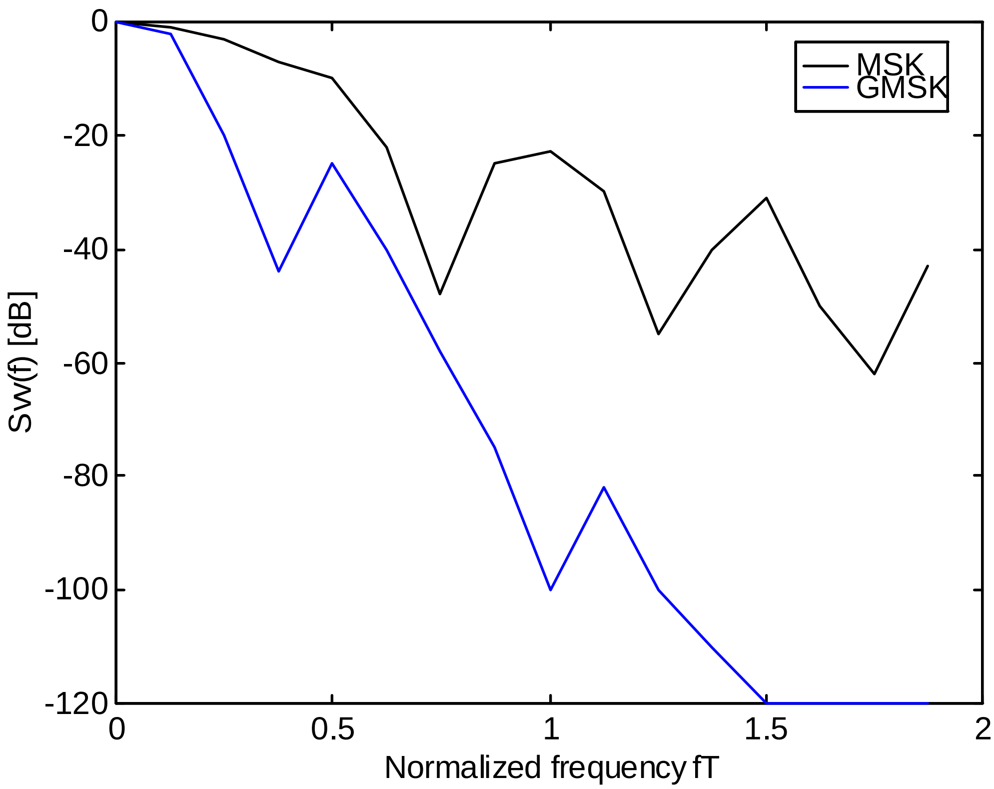

Figure 3: Comparison between MSK and GMSK PSDs

References:

Most of the material presented above is based on information provided during class lectures. Deeper description of some features is taken from

[1] 'Principles of Mobile Communications', Gordon L. Stuber, Kluwer Academic Publishers, Boston, 1996, and

[2] 'Digital Communications', John G. Proakis, McGraw-Hill, third edition

DETECTION THEORY

The purpose of this part is to revisit some of the basic concepts of detection of signals in noise, as well as the way to calculate the probability of error for various modulation schemes.

Let us recall the basic principles of any general communication system. The transmitter transmits a sequence of messages from a given alphabet $\{m_1, m_2, \ldots, m_M\}$ of size M. The messages are represented as signal waveforms $s_1(t), s_2(t), \ldots, s_M(t)$, respectively, for the purpose of transmission. Each of the waveforms has a duration T called the symbol period. One waveform is transmitted every T seconds. The transmitted waveform depends on the random sequence that is the actual information encoded for transmission. Recall that if the input sequence were known in advance, no transmission would be necessary. Therefore, when the input message m equals m_i i=0,1,...,M-1, the transmitted signal is $s_i(t)$. Thus the correspondence,

$$m = m_i \Leftrightarrow s(t) = s_i(t) \tag{18}$$

The a priori probabilities {P(m_i)} of the occurrence of each of the input messages specify the input source. A convenient way to synthesize and represent the transmitting signal waveforms is to use orthonormal signal set defined for the whole transmitting signal set {s_i(t)}. Then each and every signal from the given set {s_i(t)} will be represented as the linear combination of orthonormal waveforms as:

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad i=0,1,\dots,M-1 \tag{19}$$

where {$\phi_j(t)$} are orthonormal, by which is implied:

$$\int_{-\infty}^{\infty} \phi_j(t)\phi_k(t)dt = \begin{cases} \delta_{jk} & j = k \\ 0 & j \neq k \end{cases} \text{ for all } j \text{ and } k \ \ 0 \le j,k \le N$$

where the number N is dimension of the signal space. Note that N ≤ M.

Therefore, we can see that the signal $s_i(t)$ is completely determined by coefficients s_{ij} in equation (19) and therefore can be represented as a vector of these coefficients:

$$\mathbf{s_i} = [s_{i1}, s_{i2}, \dots, s_{im}]^T, \qquad i = 0,1,\dots,M-1$$

where

$$s_{ik} = \int_{-\infty}^{+\infty} s_i(t)\phi_k(t)dt \qquad i = 0,1,\dots,M-1$$

Usually, the signals are visualized in N-dimensional space, named signal space, where the set {ϕ_i(t)} of orthonormal functions represent the axes. Thus, {s_{ik}} are projections of the signal $s_i(t)$ onto each of the axes. The axes are, of course, mutually perpendicular.

Recall that using such a representation, the white Gaussian noise has infinite dimensionality. Fortunately, using the theorem of irrelevance, the only noise detectable and relevant for the detection of signals {s_i(t)} is the one represented by its projections onto defined orthonormal set, also called basis. Therefore, the relevant additive white Gaussian noise can also be represented as a vector of same dimension N.

Thus, the received signal vector can be represented as

$$\mathbf{x} = \mathbf{s_i} + \mathbf{w}$$

where **w** is the additive white Gaussian noise (AWGN) vector. It can also be shown that if the Gaussian noise process is of zero mean and variance $N_0/2$, also the projections will be a set of statistically independent set of Gaussian random variables with zero mean and same variance $N_0/2$. This property will significantly reduce the computational effort required for the communication system probability of false detection analysis.

Having all this in mind, the design goal is to derive the rule for the detector that by observing the received vector **x** would assign the transmitted message to it with minimum probability of false decision. Ones such a detector is designed, it is called optimum detector. Note that the optimality criterion is minimization of the probability of false detection.

Stating the problem more formally,

Given **x**, we form the set of a posterior probabilities defined as P{signal s_j was transmitted | **x**} for all signals from a given alphabet. The decision is then based on selecting the signal corresponding to maximum a posteriori probability. Such a criterion is called MAP (maximum a posteriori probability).

Then, our estimate $\hat{m} = m_i$ would be wrong with probability

$$P_e(m_i, \mathbf{x}) = P(m_i \text{ is not sent} \mid \mathbf{x}) \quad (20)$$
$$= 1 - P(m_i \text{ is sent} \mid \mathbf{x}) \quad (21)$$

Therefore, we have to maximize the probability term in equation (21) or equivalently to minimize the equation (20). Hence, the decision rule is

$$Set\ \hat{m} = m_j\ if\ P(m_j\ sent \mid \mathbf{x}) \geq P(m_k\ sent \mid \mathbf{x})\ for\ \forall j \neq k,\ k = 0,1,\ldots,M-1$$

Using Bayes rule, the a posterior probabilities can be expressed as:

$$P(m_j \mid \mathbf{x}) = \frac{P[m_j] f_{\mathbf{x}}(\mathbf{x} \mid m_j)}{f_{\mathbf{x}}(\mathbf{x})}$$

In the above equation, it is evident that the denominator does not depend on transmitted message and can therefore be neglected during detection process. A receiver that determines $\hat{m}$ by maximizing $f_x(\mathbf{x}|m_j)$ is called ML (maximum likelihood) receiver. Such a receiver is often used when the a priori set of probabilities is not known. Note that when all the transmitted messages are equally likely, the ML and MAP receivers yield the same result.

When the noise is AWGN and the source sequence and noise are statistically independent, we have that:

$$f_{\mathbf{n}}(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{N} x_j^{\,2}}$$

Also, using some probabilistic identities:

$$f_{\mathbf{x}}(\mathbf{x} \mid \mathbf{m}_i) = f_{\mathbf{n}}(\mathbf{x} - \mathbf{m}_i \mid \mathbf{m}_i) = f_{\mathbf{n}}(\mathbf{x} - \mathbf{m}_i) \text{ for all i.}$$

Therefore, the optimum receiver sets $\hat{m} = m_k$ whenever

$$P[m_i] e^{-|\mathbf{x}-\mathbf{m}_i|^2/2\sigma^2}$$

is maximum for $i=k$. Note that maximizing this expression is equivalent to finding the i that minimizes

$$|\mathbf{x} - \mathbf{m}_i|^2 - 2\sigma^2 \ln P[\mathbf{m}_i] \tag{22}$$

This function is visualized geometrically in a signal space. Note that the first term represents the square of the Euclidean distance in such a space. Whenever the input messages are equally likely, the optimum decision rule is defined as finding the closest vector to the received one in a defined signal space.

Using the above reasoning, we can divide the whole signal space into a set of regions Z_i covering the whole space and each corresponding to the message/transmitted signal vector to which each point of the region is closer than to all other points.

When the input messages are not equally likely, the region boundaries, also called decision boundaries, are appropriately modified in a way that at the boundary between two of the signal vectors, the probability of decision mistake is equal.

Before giving the actual structure of optimum receiver, we will first rearrange square term of the equation (22):

$$|\mathbf{x} - \mathbf{m}_i|^2 = \sum_{j=1}^{N} (x_j - m_{ij})^2 = |\mathbf{x}|^2 - 2\mathbf{x} \bullet \mathbf{m}_i + |\mathbf{m}_i|^2$$

Since the $|\mathbf{x}|^2$ term is independent of i the decision rule is equivalent to *maximizing* the expression:

$(\boldsymbol{x}.\boldsymbol{m}_i) + c_i$

where

$$c_i \mathrel{\hat{=}} \frac{1}{2}\left(2\sigma^2 \ln P[m_i] - |s_i|^2\right) \qquad \text{for all } i = 0,1,\ldots,M-1$$

and is usually called the *bias term*. Note that $|s_i|^2$ is the signal energy.

From the above, there are at least two possible realizations of the optimum receiver:

- **correlation receiver,** and
- **matched filter receiver**

The structure of the *correlation receiver* is depicted in Figure 4, while *matched filter receiver* is given in Figure 5.

Note that the correlation receiver first projects the received signal *x(t)* onto the orthonormal set, thus forming the received signal vector **x**, and then performs the 'dot-product' operation for all signal vectors $\mathbf{s}_i$. Finally, the bias term is added to account for different signal a priori probabilities and energies, and the largest result yields the minimum probability of error estimate of the transmitted message.

Another version of the same receiver type avoids using multiplication. Instead, it makes use of the filters matched to the orthonormal functions. By matched, we mean time reversed version of the desired signal:

$\phi_1(T-t)$ is matched to $\phi_1(t)$ where T is symbol period

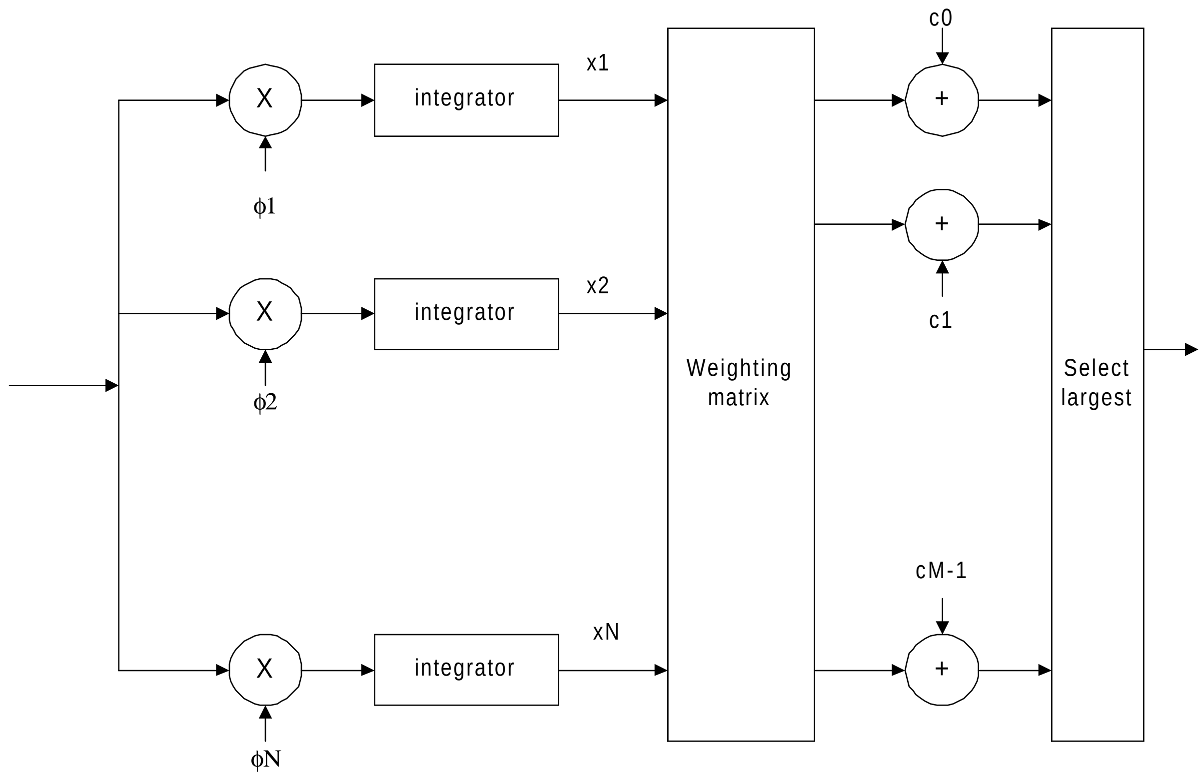

Figure 4: Correlation receiver structure

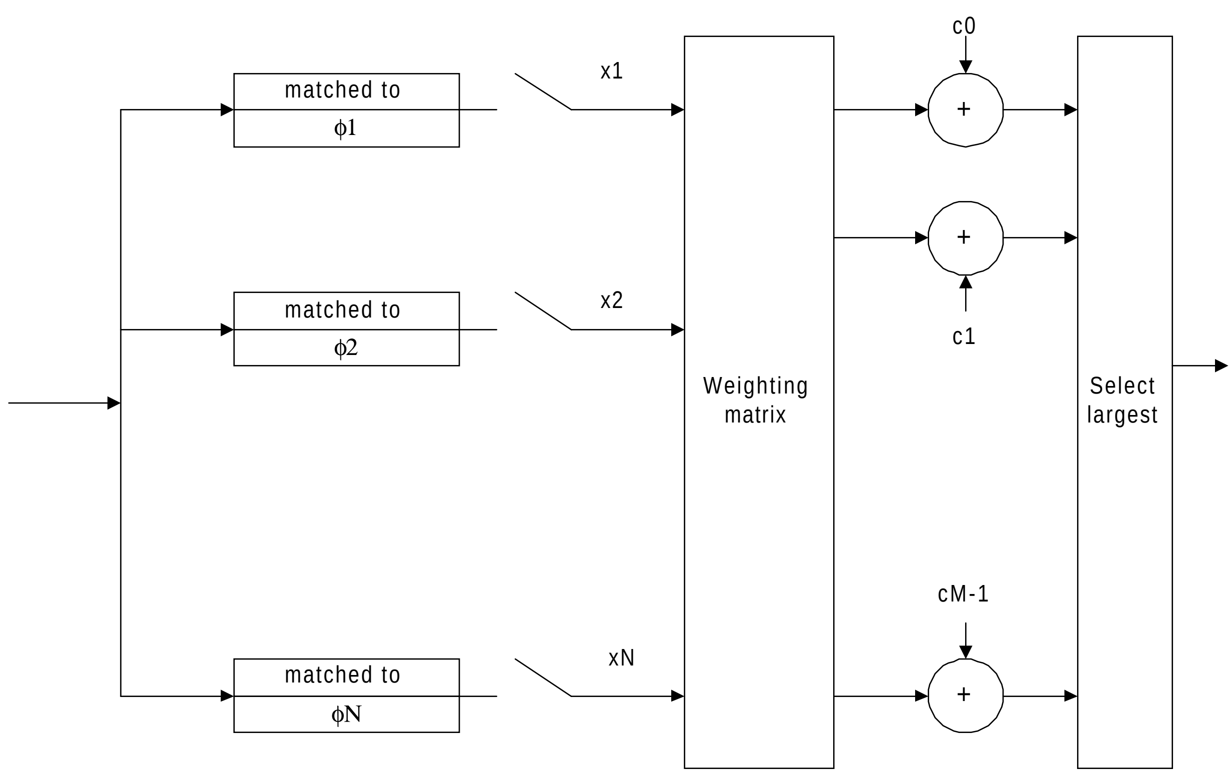

Figure 5: Matched filter receiver structure

By sampling the matched filter output at t=T, we receive the same result as with correlation receiver. The rest of the detection procedure is the same.

Namely, the filter response would be:

$$u_j(t) = \int_{-\infty}^{\infty} x(\alpha)\phi_j(T - t + \alpha)d\alpha$$

and sampled at *t=T*.

$$u_j(T) = \int_{-\infty}^{\infty} x(\alpha)\phi_j(\alpha)d\alpha = x_j$$

Another way of showing that these filters yield optimum decision with respect to error probability is showing that they also yield the maximum signal to noise ratio (SNR) reception. The formal proof is based on the Schwarz inequality and will not be given here.

Understanding all this, our final goal is to calculate the average probability of error. Recall that in general case, the error will occur if a symbol m_i is transmitted, and the received vector **x** does not lie in a region Z_i. Average probability of error is

$$P[\varepsilon] = \sum_{i=1}^{M} P[\mathbf{x}\, does\, not\, lie\, in\;\, Z_i\,, m_i\; was\, sent]$$

Assuming all the messages are equally likely, the above equation can be written as:

$$P[\varepsilon] = \frac{1}{M}\sum_{i=1}^{M} P[\mathbf{x}\, does\, not\;\, lie\, in\;\, Z_i | m_i\; sent]$$

or

$$P[\varepsilon] = 1 - \frac{1}{M}\sum_{i=1}^{M}\int_{Zi} f_{\mathbf{x}}(\mathbf{x}\,|\,m_i)d\mathbf{x}$$

where the second term corresponds to average probability of a correct decision. In a general case, the above integral is very difficult to solve, sometimes computationally intractable. In such cases we can try to estimate the bound of the performance. One of the bound is called *union bound*.

Recall from the probability theory that:

$$P(\bigcup_{k=1}^{M} A_k) \leq \sum_{k=1}^{M} P(A_k)$$

with equality when all the events A_k are mutually exclusive. Taking this concept into our detection problem, we can denote A_{ik} as event that the observation vector **x** is closer to $\mathbf{s_k}$ than to $\mathbf{s_i}$ when m_i was sent. Then, the conditional probability of error given that m_i was sent is

$$P_e(m_i) = P(\bigcup_{\substack{k=1 \\ k\neq i}}^{M} A_{ik}) \leq \sum_{\substack{k=1 \\ k\neq i}}^{M} P(A_{ik}) \quad \text{for all i=1,2,...,M}$$

where $P(A_{ik})$ is the probability that the vector **x** is closer to $\mathbf{s_k}$ than to $\mathbf{s_i}$. This will occur when the projection of the noise onto a line that joins $\mathbf{s_k}$ and $\mathbf{s_i}$ is greater than half the distance, assuming that the messages are equally likely. Recall that the projection of the AWGN yields the set of statistically independent random variables with equal mean and variance. Therefore, we are only interested in the noise component that lies on that line. Developing this, we have

$$P_{ik} = \int_{d_{ik}/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{u^2}{N_0}} du = \frac{1}{2} erfc(\frac{d_{ik}}{2\sqrt{N_0}})$$

where d_{ik} is the distance between vectors $\mathbf{s_k}$ and $\mathbf{s_i}$, and *erfc()* is a standard complementary error function. Using the union bound:

$$P_e(m_i) \leq \frac{1}{2} \sum_{\substack{k=1 \\ k\neq i}}^{M} erfc(\frac{d_{ik}}{2\sqrt{N_0}})$$

Then, using the symmetry and assuming equally likely input messages, the total average probability of error can be expressed as:

$$P_e \le \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{M} erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) = \sum_{\substack{k=1\\k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

We shall now derive the average probability of error for some generally used modulation schemes.

Coherent Binary Phase Shift Keying (CBPSK)

With M = 2, the mapping of source symbols m_i to the transmitting waveforms $s_i(t)$, i = 1, 2 is given by

$$1 \to s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos 2\pi f_c t \qquad \text{for } 0 \le t \le T_b$$

$$0 \to s_2(t) = -\sqrt{\frac{2E_b}{T_b}}\cos 2\pi f_c t \qquad \text{for } 0 \le t \le T_b$$

where f_c is the carrier frequency, much greater than $1/T_b$, and E_b is energy per transmitted bit. The signal space looks like in Figure 6.

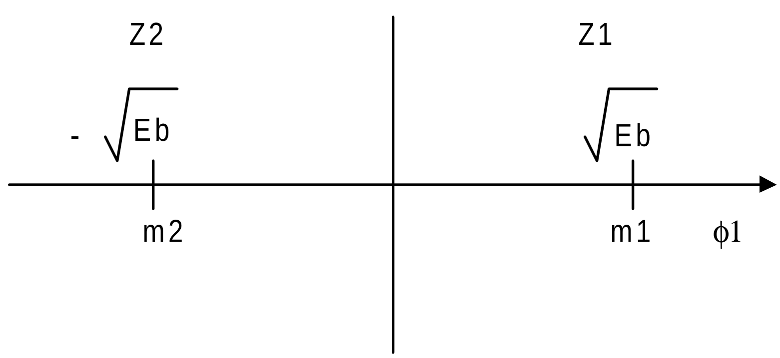

Figure 6: The signal space with decision regions for CBPSK

The basis function is: $\phi 1(t) = \sqrt{\frac{2}{T_b}}\cos 2\pi f_c t$

The appropriate decision rule for equally likely messages is: Choose m_1 when **x** is greater than 0 (in Z_1), otherwise choose m_2.

Then,

$$x(t) = \int_0^{T_b} x(t)\phi_1(t)dt = \int_0^{T_b} (s_i(t) + w(t))\phi_1(t)dt$$

The conditional probability of error when m_1 is sent is then

$$P_e = P\{\mathbf{x} > 0 \mid m_1\, is\, sent\} = \frac{1}{2} erfc(\sqrt{\frac{E_b}{N_0}}) = Q(\sqrt{\frac{2E_b}{N_0}})$$

Assuming equally likely messages, that is also the total average probability of error for CBPSK system.

Coherent Quadrature Phase Shift Keying (CQPSK)

Transmitting signal is given by:

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + (2i-1)\frac{\pi}{4}\right) & i = 1,2,3,4 \\ 0 & otherwise \end{cases} \quad \text{for } 0 \le t \le T$$

where f_c is the carrier frequency, much greater than 1/T, T is symbol period, and E is energy per transmitted symbol. The signal space looks like in Figure 7.

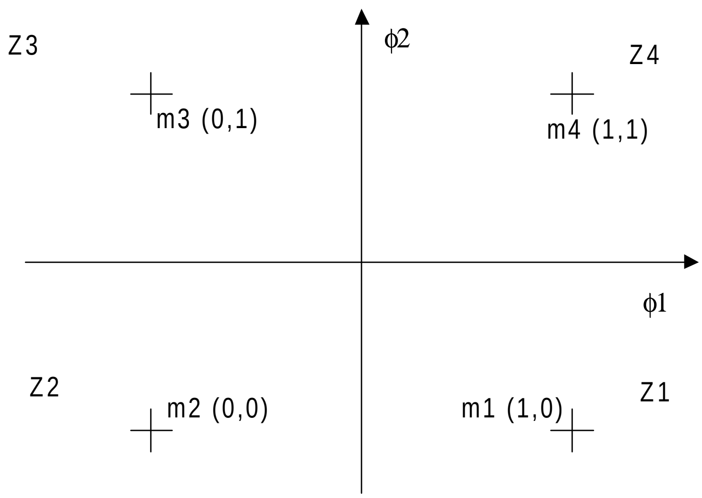

Figure 7: The signal space with decision regions for CQPSK

The basis functions are:

$$\phi 1(t)=\sqrt{\frac{2}{T}}\cos 2\pi f_c t \qquad \text{and} \qquad \phi 2(t)=\sqrt{\frac{2}{T}}\sin 2\pi f_c t$$

The appropriate decision rule for equally likely messages is: Choose m_i when **x** is lies in Z_i, where the regions are limited by axes.

Then,

$$\mathbf{s}_i=\begin{bmatrix}\mathbf{s}_{i1}\\ \mathbf{s}_{i2}\end{bmatrix}=\begin{bmatrix}\sqrt{E}\cos((2i-1)\frac{\pi}{4})\\ -\sqrt{E}\sin((2i-1)\frac{\pi}{4})\end{bmatrix} \qquad \text{for i=1,2,3,4}$$

Then, the received vector **x** is:

$$\mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=\begin{bmatrix}\pm\sqrt{\frac{E}{2}}+w_1\\ \mp\sqrt{\frac{E}{2}}+w_2\end{bmatrix}$$

To calculate the average probability of error, note that QPSK system is equivalent to two coherent BPSK systems that modulate two orthogonal carriers at the same frequency f_c. Note that E is the symbol energy and that each symbol represents two bits.

Therefore, the average probability of a symbol error for CQPSK system is:

$$P'=\frac{1}{2}erfc(\sqrt{\frac{E_s}{2N_0}})$$

Then, expressing the probability of correct decision as $P_c=(1-P_e)^2$ we can calculate the total average symbol error probability as:

$$P_{se}=1-P_c=erfc(\sqrt{\frac{E_s}{2N_0}})-\frac{1}{4}erfc^2\left(\sqrt{\frac{E_s}{2N_0}}\right)$$

For large SNR, this can be approximated as:

$$P_{se} = erfc(\sqrt{\frac{E_s}{2N_0}})$$

When the Gray encoding is used, as usually is, then one symbol error results most often in only one bit error. Therefore, the bit error probability for CQPSK system will approximately be half the symbol error:

$$P_e \approx 0.5\, P_{se}$$

Knowing that there are two bits per symbol and that the E_s is symbol energy = $2*E_b$, we can express total average bit error probability for CQPSK system as:

$$P_e = \frac{1}{2} erfc(\sqrt{\frac{E_b}{N_0}}) = Q(\sqrt{\frac{2E_b}{N_0}})$$

Note that for the same SNR, CBPSK and CQPSK systems have the same probability of error, but with CQPSK we can transmit two times more data than with CBPSK.

M – ary Phase Shift Keying (M-PSK)

Transmitting signal is given by:

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos(2\pi f_c t + (i-1)\frac{2\pi}{M}) \qquad \text{for } 0 \le t \le T_s$$

where f_c is the carrier frequency, much greater than $1/T_s$, T_s is symbol period, and E_s is energy per transmitted symbol.

Note that $E_s = E_b \log_2 M$, $T_s = T_b \log_2 M$

Using the union bound, the average bit error probability is given as:

$$P_e \le \sum_{\substack{k=1 \\ k \ne i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

Note that as M increases, the distances among signal vectors decrease, thus the performance deteriorates.

Coherent Binary Frequency Shift Keying (CBFSK)

For coherent binary frequency shift keying, we have transmitting signals:

$$1 \rightarrow s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_1 t \qquad \text{for } 0 \le t \le T_b$$

$$0 \rightarrow s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_2 t \qquad \text{for } 0 \le t \le T_b$$

where the frequencies are chosen as: $f_i = \frac{n_c + i}{T_b}$ with i=1,2 and n_c being some integer.

The basis functions are:

$$\phi 1(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_1 t \qquad \text{and} \qquad \phi 2(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_2 t$$

The transmitted signal vectors can then be represented as:

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \qquad and\, \mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

The signal space representation is shown in Figure 8.

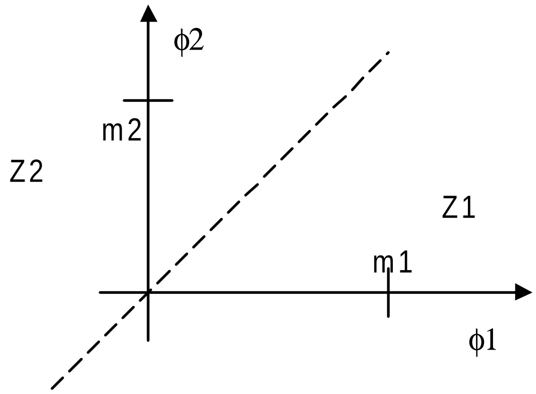

Figure 8: Signal space representation for CBFSK

The decision rule is: Choose *m1* when $x_1 > x_2$, otherwise choose *m2*.

The probability of error is given as:

$$P_e = \frac{1}{2} erfc(\sqrt{\frac{E_b}{2N_0}}) = Q(\sqrt{\frac{E_b}{N_0}})$$

The comparison among several M-PSK modulation schemes with respect to bandwidth efficiency and power efficiency for the required BER of 10^{-6} is given in the following table.

M	2	4	8	16	32	64
$\eta_p = E_b/N_0$	10.5	10.5	14	18.5	23.4	28.5
$\eta_b = R_b/B$	0.5	1	1.5	2	2.5	3

Table 1: The performance comparison among M-PSK modulation schemes for the required BER of 10^{-6}

Minimum Shift Keying (MSK)

Recall that the MSK signal can be represented as:

$$s(t) = A\cos\left[\left(2\pi f_c + \frac{\pi x_k}{2T}\right)t + \frac{\pi}{2}\sum_{n=-\infty}^{k-1} x_n - \frac{\pi}{2}x_n\right]$$

If we consider interval $0 \le t \le T_b$ and denote $A = \sqrt{\frac{2E_b}{T_b}}$, then:

$$s(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta(0)) & for\,symbol\,1 \\ \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_2 t + \theta(0)) & for\,symbol\,0 \end{cases}$$

where the phase term depends on the accumulated excess phase. Using phase trellis and e.g. Viterbi decoding scheme, it can be shown that for high SNR, the average bit error probability for MSK can be approximated as:

$$P_e \approx Q(\sqrt{\frac{2E_b}{N_0}})$$

References:

The material presented above is largely based on information provided during class lectures. Additional information is taken from:

[1] 'Principles of Mobile Communications', Gordon L. Stuber, Kluwer Academic Publishers, Boston, 1996,
[2] 'Digital Communications', John G. Proakis, McGraw-Hill, third edition
[3] 'Principles of Communication Engineering', Wozencraft, Jacobs, Waveland Press, 1990.