Wireless Communications Technologies

Course No: 16:332:546

Homework 1

1. (a) The characteristic function of $X_{i,n}$ is

   $M_{X_{i,n}}(u) = E[\exp(juX_{i,n})] = (1-\lambda/n) + \exp(ju)\lambda/n = 1 + \frac{\lambda}{n}[\exp(ju)-1]$

   Therefore,

   $M_{Y_n}(u) = \{1 + \frac{\lambda}{n}[\exp(ju)-1]\}^n$

   (b) $\lim_{n\to\infty} M_{Y_n}(u) = \exp(\lambda(\exp(ju)-1))$,

   which implies $\lim_{n\to\infty} Y_n$ is a Poisson random variable with mean and variance λ. (This follows from the fact that the mapping from a characteristic function to a distribution is 1:1)

2. $E[Y_t] = \mu D$

   By definition $\phi_Y(t,s) = E[(X_{t+D} - X_t)(X_{s+D} - X_s)]$. Therefore,

   $\phi_Y(t,s) = \sigma^2[\min(t+D, s+D) - \min(t+D, s) - \min(t, s+D) + \min(t,s)] + \mu^2[(t+D)(s+D) - (t+D)s - t(s+D) + ts]$

   There are 2 cases:

   Case 1: $\mid t-s \mid \leq D$

   $\phi_Y(t,s) = \sigma^2[D - \mid t-s \mid] + \mu^2 D^2$

   Case 2: $\mid t-s \mid > D$

   $\phi_Y(t,s) = \mu^2 D^2$

   From Case 1 and Case 2, it is clear that Y_t is wide-sense-stationary. Further, since Y_t is Gaussian, it is strictly stationary!

3. The analog signal is sampled at $f_s = 8\,Khz$. Each sample is quantized with $L = 64$ levels of representation. Therefore the number of bits R required to represent each sample is

   $$R = \log_2 L = 6\,\text{bits}$$

   The total bit rate after sampling and quantization is $f_s \times R\ Kbps$.

   The minimum transmission bandwidth required W is given as $W = \frac{1}{2T}$, where T is the symbol duration of the M-ary PAM system.

   (a) $M = 2$

   For $M = 2$ amplitude levels, each pulse can represent $\log_2 M = \log_2 2 = 1$ bit. Therefore,

   $$T = \frac{1}{f_s R}\log_2 M = \frac{1}{f_s R}$$

$\Rightarrow W = f_s R/2 = 48/2\, Khz = 24\, Khz$

(b) $M = 4$

For $M = 4$ amplitude levels, each pulse can represent $\log_2 M = \log_2 4 = 2$ bits. Therefore,

$$T = \frac{1}{f_s R} \log_2 M = \frac{1}{f_s R} \times 2$$

$\Rightarrow W = f_s R/4 = 48/4\, Khz = 12\, Khz$

4. A bit 1 is represented by a pulse of height A for a duration of 1 second and a bit 0 is represented by sending no pulse for a duration of 1 second. The signals are transmitted over a AWGN channel with zero mean and power spectral density 1/2. Let y denote the output of the integrator in Figure 1.

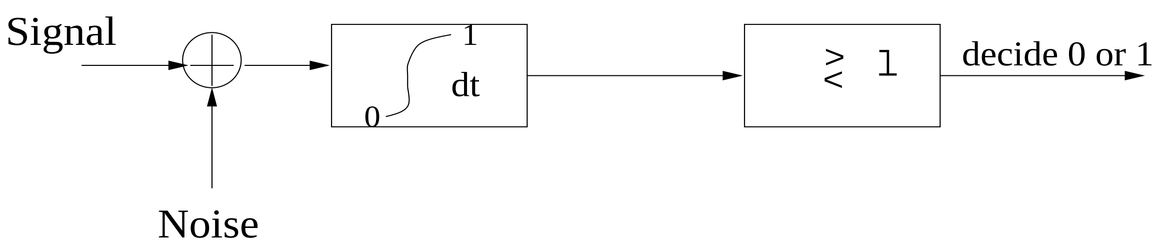

Figure 1: Receiver for the PCM System with On-off Keying

(a) For equiprobable bit-transmission, $p_0 = p_1 = 1/2$. To find the optimum threshold λ that minimizes the probability of error, we need to solve the following equation

$$\frac{p_0}{p_1} = 1 = \frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda_{opt}|0)} \tag{1}$$

Let us first find the density functions $f_Y(y|1)$ and $f_Y(y|0)$

When a 1 is transmitted

$$Y = A + \int_0^1 w(t)dt$$

It follows that y is a Gaussian random variable with $E[Y|1] = A$, and variance

$$\sigma^2_{Y|1} = E[\int_0^1 \int_0^1 w(t)w(u)dtdu] = \int_0^1 \int_0^1 \frac{1}{2}\delta(t-u)dtdu = \frac{1}{2}$$

Therefore

$$f_Y(y|1) = \frac{1}{\sqrt{\pi}} \exp(-(y-A)^2) \tag{2}$$

Similarly, when a 0 is transmitted

$$Y = 0 + \int_0^1 w(t)dt,$$

and it follows that y is a Gaussian random variable with $E[Y|0] = 0$, and variance

$$\sigma^2_{Y|0} = E[\int_0^1 \int_0^1 w(t)w(u)dtdu] = \int_0^1 \int_0^1 \frac{1}{2}\delta(t-u)dtdu = \frac{1}{2}$$

Therefore

$$f_Y(y|0) = \frac{1}{\sqrt{\pi}} \exp(-y^2) \tag{3}$$

Using equations (2) and (3) in equation (1), we get

$$1 = \frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda_{opt}|0)} = \frac{\exp(-(\lambda_{opt} - A)^2)}{\exp(-\lambda^2_{opt})}$$

Taking log on both sides and rearranging, we get

$$\lambda^2_{opt} = (\lambda_{opt} - A)^2$$

$\Rightarrow \lambda_{opt} = A/2$.

I guess you could have guessed this answer knowing that either A or 0 was being transmitted with equal probability in AWGN of zero mean!

(b) Using the threshold in part (a), i.e., $\lambda = A/2$, we can evaluate the average probability of error for this receiver in terms of the the complementary error function erfc(x) as follows :

Consider a zero being transmitted, then the conditional probability of making an error is

$$P_{e0} = P(y > \frac{A}{2}|0) = \int_{A/2}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-y^2) = \frac{1}{2}\text{erfc}(\frac{\text{A}}{2})$$

By symmetry it follows that $P_{e1} = P_{e0} \Rightarrow$

$$P_e = P_{e1} = P_{e0} = \frac{1}{2}\text{erfc}(\frac{\text{A}}{2})$$

5. The channel bandwidth is given to be $B = 60\,KHz$ and the bit rate is $R_b = 100\,Kbps$. The bit duration is therefore given as $T_b = 1/R_b = 10\,\mu sec$.

   The signal bandwidth can be found as $W = \frac{1}{2T_b} = 50\,kHz$

   Therefore, the raised cosine pulse should be designed such that its *rolloff factor* α satisfies

$$B = W(1 + \alpha)$$

   $\Rightarrow \alpha = 0.2$

6. Consider a set of orthonormal basis functions $\{\phi_j(t)\}_{j=1}^N$. Let $w(t)$ be an AWGN process of zero mean and p.s.d. $\frac{N_0}{2}$.

We need to show that the sequence $\{w_j\}_{j=1}^N$ are i.i.d. Gaussian random variables, where

$$w_j = \int_0^T w(t)\phi_j(t)dt, \;\; j = 1, \cdots, N.$$

Since $w(t)$ is a Gaussian process, it follows that w_j is a Gaussian random variable. Further, $E[w_j] = 0$, since $w(t)$ is zero mean.

Consider the covariance function

$$Cov(w_j\, w_k) = E[w_j\, w_k] = E[\int_0^T w(t)\phi_j(t)dt \; \int_0^T w(t)\phi_k(t)dt]$$

Rearranging the integrals $\Rightarrow$

$$Cov(w_j\, w_k) = E[\int_0^T \int_0^T w(t)\phi_j(t)w(u)\phi_k(u)dtdu]$$

Taking the expectation inside the integral $\Rightarrow$

$$Cov(w_j\, w_k) = \int_0^T \int_0^T \phi_j(t)\phi_k(u)E[w(t)w(u)]dtdu$$

But $E[w(t)w(u)] = \frac{N_0}{2}\delta(t-u) \Rightarrow$

$$Cov(w_j\, w_k) = \frac{N_0}{2}\int_0^T \phi_j(t)\phi_k(t)dt = 0$$

$\Rightarrow w_j$ and w_k are uncorrelated.

When $j = k$, $Cov(w_j\, w_j) = Var(w_j) = \frac{N_0}{2} \Rightarrow$ the random variables w_j have the same variance as well.

Therefore, the sequence $\{w_j\}_{j=1}^N$ are uncorrelated and identically distributed. Since they are Gaussian, it follows that they are also independent.

7. We first observe that the signals $\{s_i(t)\}$ $i = 1, 2, 3$ are linearly independent.

   The energy of signal $s_1(t)$ is given as

$$E_1 = \int_0^T s_1^2(t)dt = 4,$$

where $T = 3$. Therefore, the first basis function is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Based on the definition of the coefficients as

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \tag{4}$$

we can find that $s_{21} = -4$.

Based on definition of the function $g_i(t)$ as

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t), \tag{5}$$

we can evaluate $g_2(t)$ as

$$g_2(t) = \begin{cases} -4, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

The second basis function is now given as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} = \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Using equation (4), we can now compute

$$s_{31} = 3, \;\; s_{32} = -3$$

Using the above coefficients in (5), we get

$$g_3(t) = \begin{cases} 3, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the third basis function is given as

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}} = \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

We can write the signals in terms of the basis functions as

$$s_1(t) = 2\phi_1(t)$$

$$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$$

$$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$$

8. Consider the set of signals $\{s_i(t)\}_{i=1}^{i=4}$, where the signal $s_i(t)$ is of the form

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi\frac{t}{T} + i\frac{\pi}{4}), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Observe that using the cosine formula $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, we can write each of the above signals as

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}}[\cos(2\pi\frac{t}{T})\cos(i\frac{\pi}{4}) - \sin(2\pi\frac{t}{T})\sin(i\frac{\pi}{4})], & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Therefore each signal can be written as a weighted sum of the two functions $\cos(2\pi\frac{t}{T})$ and $\sin(2\pi\frac{t}{T})$. Do these two functions make an orthonormal basis ?

They do if we choose $\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi\frac{t}{T})$ and $\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi\frac{t}{T})$, since we can easily verify that

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1, & \text{if} i = j \\ 0, & \text{if} i \neq j \end{cases}$$

Therefore, each of the signals can now be written as

$$s_i(t) = \begin{cases} \sqrt{E}\cos(i\frac{\pi}{4})\ \phi_1(t) \ - \ \sqrt{E}\sin(i\frac{\pi}{4})\ \phi_2(t), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Therefore the coefficients in the expansion are

$$s_{11} = \sqrt{E}\cos(\frac{\pi}{4}) = \sqrt{E/2}, \quad s_{12} = -\sqrt{E}\sin(\frac{\pi}{4}) = -\sqrt{E/2}$$

$$s_{21} = \sqrt{E}\cos(\frac{2\pi}{4}) = 0, \quad s_{22} = -\sqrt{E}\sin(\frac{2\pi}{4}) = -\sqrt{E}$$

$$s_{31} = \sqrt{E}\cos(\frac{3\pi}{4}) = -\sqrt{E/2}, \quad s_{32} = -\sqrt{E}\sin(\frac{3\pi}{4}) = -\sqrt{E/2}$$

$$s_{41} = \sqrt{E}\cos(\frac{4\pi}{4}) = -\sqrt{E}, \quad s_{42} = -\sqrt{E}\sin(\frac{4\pi}{4}) = 0$$