A Framework for Dynamic Spectrum Sharing between Cognitive Radios

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***Abstract*— We consider a cognitive radio system like the future 802.22 networks where license-exempt service providers (SPs) will share a fixed spectrum in a non-interference basis to each other and also to the licensed users in that spectrum. The percentage of spectrum utilized by one SP depends on how many users it is serving and how much spectrum each user application demands. We assume that an user can obtain service from all the SPs. The quality of service depends on system parameters such as number of users and SPs, the channel conditions between the users and SPs and the total power available at each user. We adopt an user utility maximization framework to analyze this system. Given the user utility functions, and the above mentioned system parameters we derive optimal values of spectrum that the users should obtain from the SPs. We also introduce the notion of spectrum price and use it to demonstrate several key results about spectrum allocation. The spectrum price proves to be the regulatory mechanism that brings about coordination amongst the SPs with minimal control messaging. Our approach thus strikes a balance between a centralized network and a fully uncoordinated open access network.**

I. Introduction

In the 1950s, the FCC in USA sold licenses for 330 MHz of spectrum for UHF television. This experiment never took off leading to considerable bands of unused spectrum between VHF and UHF broadcast channels from 54 to 865 MHz for example in New York city. Recently in October 2004, the IEEE set up a working group to develop the 802.22 standard that would employ these unused spectrum to offer wireless broadband services to areas not well served by alternatives such as cable or DSL [1].

In the 802.22 draft it has been decided that fixed wireless access will be provided in these bands [5] by professionally installed Wireless Regional Area Network (WRAN) base stations to WRAN user terminals. A service provider (SP) operating a base station will not have to pay any licensing fees. Since the spectrum within the 54 to 865 MHz range where the TV broadcasters operate varies within location, a WRAN SP would have to sense the presence of digital televisions (DTVs) and NTSC and also other WRAN SPs before deciding in which part of the spectrum to operate. Hence the term *cognitive radio* is often used to denote the WRAN base stations and user devices. Fundamental issues associated with sensing can be found in [9].

The physical layer of 802.22 cognitive systems is based on OFDMA [12]. The task of spectrum sharing is simplified as spectrum utilized by an SP directly translates to a set of OFDM subchannels.

Since the 802.22 systems are unlicensed, one simple system implementation would be to deploy the WRAN SPs independently with no coordination amongst SPs and no central control over them. Using the sensing mechanisms [9] and how many users they have to serve, the SPs would deduce how much spectrum to use. Indeed the alternative principle of strict centralized control might work well for licensed networks like cellular but goes against the principles of a cognitive radio network. However as pointed out in [3], [4] a purely uncoordinated system, though simple to implement, will suffer from interference and quality of service guarantees will be hard to offer. They argue that a pragmatic system should have some simple coordination amongst the SPs.

A. Our Contribution

Given the intense interest around dynamic spectrum allocation for cognitive radios, it becomes important to study this problem from an analytical framework. In this paper we analyze a two tiered spectrum allocation scheme as shown in Figure 1. We allow for simple SP coordination to share spectrum. This is facilitated by a spectrum clearing house (SCH), akin to an FCC-controlled regional spectrum broker [3]. The users could potentially obtain spectrum from all the SPs. Since WRAN users are assumed not to interfere with each other, we assume that they obtain non-overlapping chunks of spectrum from the SPs. We assume that each user application has an associated utility expression as a function of spectrum obtained and adopt an utility maximization framework [8] to analyze the system. Given user utility functions, channel coefficients between users and SPs and user power constraints, our aim is to derive how much spectrum should a user obtain from an SP and what power should he allocate for sending his information to the SPs. We introduce the notion of *spectrum price* and show that the marginal increase in complexity due to coordination is placed only on the SPs and the user operation still turns out to be distributed. For utility functions that are linear in transmission rate we derive several key results about the allocated spectrum.

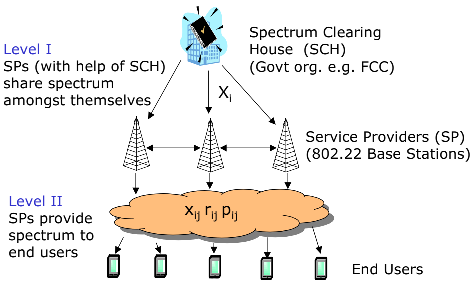

Fig. 1. The network topology

B. Related Work in Multi-Tiered Spectrum Allocation

Buddhikot et. al. [3], [4] propose two and three tiered hierarchical spectrum allocation systems. They introduce the notion of Coordinated Access Bands of Spectrum which are dynamically assigned to various SPs by a central entity called the Regional Spectrum Information Manager. They focus their attention on the network architecture and protocol design for the system. The two tiered model is also considered by the authors in [7] who take a mixed open access/property rights regime where two service providers dynamically compete for customers as well as portions of available spectrum from a central Spectrum Policy Server. As noted in [4], a multi-tiered model enables unbundling of providers and creates competition.

II. System Model

Consider the network topology shown in Figure 1. Assume that there are N SPs and L end users and one central Spectrum Clearing House (SCH). Based on the demand for spectrum, Service Provider i provides X_i units to the L users or a subset of them. In the most general model, each user can obtain spectrum from each SP. Let x_{ij} be the amount of spectrum obtained by user j from SP i. In the subsequent communication process, user j transmits his data to SP i over this spectrum at rate r_{ij} and with power p_{ij}. For each user, there is a constraint on the total transmit power. The channel between SP i and user j is characterized by the link gain coefficient h_{ij}, which remains constant during the period of spectrum allocation and subsequent transmission to the SP. The coefficients h_{ij} are assumed to be known at both the users and the SPs. We also assume that the background additive Gaussian noise is of unit power spectral density. The maximum value of r_{ij} is decided by x_{ij}, p_{ij} and h_{ij} as per the Shannon capacity expression $x_{ij}\log(1+h_{ij}p_{ij}/x_{ij})$ scaled by a term η_i which is the fraction of the Shannon capacity that can be reliably guaranteed by SP i to an user. We call this the SP efficiency term and introduce it to differentiate between SPs in terms of offered QoS. A possible example could be SP i, who has invested more in deploying a better decoder (a Turbo decoder with more iterations or better interleaver design) has a higher η_i than an SP with a conventional Viterbi decoder, with the encoder being same in both cases. Thus the total rate at which user j can transmit reliably is thus $R_j = \sum_{i=1}^{N} r_{ij}$. We assume that there is a utility function $U_j(R_j)$ associated with user j which is concave and increasing in R_j [11]. The user welfare optimization problem is thus,

$$\max_{x_{ij},p_{ij},X_i} \sum_{j=1}^{L} U_j\left(\sum_{i=1}^{N} r_{ij}\right) \tag{1}$$

$$\text{s.t. } r_{ij} = \eta_i x_{ij}\log\left(1+\frac{h_{ij}p_{ij}}{x_{ij}}\right) \tag{2}$$

$$\sum_{j=1}^{L} x_{ij} \le X_i,\ 1 \le i \le N \tag{3}$$

$$\sum_{i=1}^{N} p_{ij} \le P_j,\ 1 \le j \le L \tag{4}$$

$$\sum_{i=1}^{N} X_i \le C \tag{5}$$

$$x_{ij}, p_{ij} \ge 0,\ 1 \le i \le N,\ 1 \le j \le L \tag{6}$$

$$X_i \ge 0,\ 1 \le i \le N. \tag{7}$$

As shown in (3), X_i is the spectrum utilized by SP i which is equal to the spectrum it has to allocate to the users. User j transmits with power p_{ij} to SP i and as (4) shows there is a constraint P_j on the total transmit power. The total amount of available spectrum for all the unlicensed devices is C.

A. Distributed Solution and Pricing

In this section we show that the spectrum allocation problem (1) - (7) is distributed at the user end. For this we form the partial Lagrangian $\mathcal{L}$ [2] by relaxing the constraints (3) and (5) in the objective function to obtain,

$$\mathcal{L}(x_{ij},p_{ij},X_i,\boldsymbol{\lambda},\mu) = \sum_{j=1}^{L} U_j\left(\sum_{i=1}^{N} r_{ij}\right) + \sum_{i=1}^{N}\lambda_i\left(X_i - \sum_{j=1}^{L} x_{ij}\right) + \mu\left(C - \sum_{i=1}^{N} X_i\right),$$

where $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_N]^T$. The dual $g(\boldsymbol{\lambda},\mu)$ is,

$$g(\boldsymbol{\lambda},\mu) = \max_{x_{ij},p_{ij},X_i} \mathcal{L}(x_{ij},p_{ij},X_i,\boldsymbol{\lambda},\mu) \tag{8}$$

$$\text{s.t. } \sum_{i=1}^{N} p_{ij} \le P_j,\ 1 \le j \le L \tag{9}$$

$$x_{ij}, p_{ij}, X_i \ge 0. \tag{10}$$

From $g(\boldsymbol{\lambda},\mu)$ we see that the optimization in (1) - (7) decomposes into separate optimization problems [2] for the users and the SPs. The optimization subproblem for user j is,

$$f_j^*(\boldsymbol{\lambda}) = \max_{x_{ij}, p_{ij}} U_j \left(\sum_{i=1}^{N} r_{ij} \right) - \sum_{i=1}^{N} \lambda_i x_{ij} \tag{11}$$

$$\text{s.t. } \sum_{i=1}^{N} p_{ij} \leq P_j \tag{12}$$

$$r_{ij} = \eta_i x_{ij} \log\left(1 + \frac{h_{ij} p_{ij}}{x_{ij}}\right) \tag{13}$$

$$x_{ij}, p_{ij} \geq 0, \ 1 \leq i \leq N. \tag{14}$$

The multiplier λ_i can be interpreted as the *shadow price* or spectrum price charged by SP i. Thus we obtain a distributed spectrum allocation procedure at the user.

The optimization subproblem solved by SP i for allocation of spectrum is

$$s_i^*(\lambda_i, \mu) = \max_{X_i} \lambda_i X_i - \mu X_i \tag{15}$$

$$\text{s.t. } X_i \geq 0, \tag{16}$$

and the master problem for evaluating the spectrum price, solved jointly by all the SPs with facilitation from the SCH is minimizing the dual $g(\boldsymbol{\lambda}, \mu) = \sum_{j=1}^{L} f_j^*(\boldsymbol{\lambda}) + \sum_{i=1}^{N} s_i^*(\lambda_i, \mu) + \mu C$,

$$\min_{\boldsymbol{\lambda}, \mu} \sum_{j=1}^{L} f_j^*(\boldsymbol{\lambda}) + \sum_{i=1}^{N} s_i^*(\lambda_i, \mu) + \mu C \tag{17}$$

$$\text{s.t. } \lambda_i \geq 0, \mu \geq 0. \tag{18}$$

Solving this problem thus gives the user-SP spectrum price λ_i. The notion of spectrum price is the key to our work. In our system, first all SPs jointly solve for the spectrum price as shown in (17) (as opposed to joint solving for every user's spectrum allocation in case of a centralized network). Next given the spectrum price, the spectrum allocation problem for the users becomes distributed as in (11) to (14). Thus our approach strikes a balance between centralized and fully uncoordinated networks.

For sake of clarity in the rest of the paper, we reiterate that the terms *payment* or *price* of spectrum means the willingness of users to obtain spectrum and not actual dollar price. As shown in Section II-A that the prices are the Lagrange multipliers associated with the total spectrum constraints of the optimization problem (1) to (7). They are also called shadow prices [8].

We state and prove an important lemma about spectrum prices.

Lemma 1: The spectrum prices from all the SPs are same.
Proof: Consider the partial Lagrangian of the optimization problem as given in (1) - (7). The stationary conditions w.r.t. X_i yields,

$$\frac{\partial \mathcal{L}}{\partial X_i} = \lambda_i - \mu \begin{cases} = 0 & \text{if } X_i > 0. \\ < 0 & \text{if } X_i = 0. \end{cases} \tag{19}$$

If an SP utilizes non-zero spectrum then $X_i > 0$ and thus $\lambda_i = \mu$. Hence proved. ■

To understand this recall that spectrum price is an indication of the user's willingness to obtain spectrum. This depends upon how much the user's utility is increased as a result of obtaining that spectrum. Consider $\eta_i = 1$ and $U_j(R_j) = R_j = \sum_{i=1}^{N} r_{ij}^{\max}$ for all i, where $r_{ij}^{\max} = r_{ij}$ as in (2) for $\eta_i = 1$. For identical channel conditions, a unit of spectrum obtained from all SPs yield the same utility to a user as per (2) and this is why spectrum prices from all SPs are same. If η_is are different and $U_j(R_j) = R_j = \sum_{i=1}^{N} r_{ij}$, a better way to interpret the price is to define a quantity called the *effective spectrum price*. By dividing (11) by η_i we obtain a new optimization problem where objective is $U_j(R_j)/\eta_i = R_j/\eta_i = \sum_{i=1}^{N} r_{ij}^{\max}$ as before but SP i now charges price $\lambda_i/\eta_i = \mu/\eta_i$. We note that this problem has the same solution as the original one. We define μ/η_i as the effective spectrum price. Thus the effective price of spectrum reduces with increasing SP efficiency thus attracting more users to an efficient SP.

Using the fact $\lambda_i = \mu$ we revisit (15) and (17) and note that $s_i^*(\boldsymbol{\lambda}, \mu) = 0$ for all i.

B. $U_j(R_j) = R_j$

In this work we mostly deal with $U_j(R_j) = R_j$. Extending notion of spectrum price for more general concave utility functions is a scope for future work. For $U_j(R_j) = R_j$ the user optimization problem (11) - (14) becomes,

$$\max_{x_{ij}, p_{ij}} \sum_{i=1}^{N} \eta_i x_{ij} \log\left(1 + \frac{h_{ij} p_{ij}}{x_{ij}}\right) - \sum_{i=1}^{N} \mu x_{ij} \tag{20}$$

$$\text{s.t. } \sum_{i=1}^{N} p_{ij} \leq P_j \tag{21}$$

$$x_{ij}, p_{ij} \geq 0, \ 1 \leq i \leq N, \tag{22}$$

where we have used the fact that $\lambda_i = \mu$ for all i from Lemma 1. Analyzing $U_j(R_j) = R_j$ is important as,

- It gives the capacity of the user-SP channel and also optimum power and spectrum allocations for achieving capacity.

Before proceeding to obtain a solution let us examine the nature of this problem. Let us assume that user j obtained spectrum chunks x_{ij} from all SPs and the optimization was done only over the power variables p_{ij}. Then the solution would have been the water-filling method [6]. In this method it is possible that out of N SPs only $M < N$ SPs would have been engaged in power allocation by user j. However when the spectrum chunk variables x_{ij} are also included in the optimization, as in (20) and (21), a waterfilling approach is not optimal as the user would still see the spectrum price for purchasing the $N-M$ spectrum chunks, though he is obtaining no utility from them. To arrive at the optimal solution we first write the Lagrangian for the problem (20) - (22),

$$\mathcal{L}_j = \sum_{i=1}^{N} \eta_i x_{ij} \log\left(1 + \frac{h_{ij} p_{ij}}{x_{ij}}\right) - \sum_{i=1}^{N} \mu x_{ij} + \gamma_j \left(P_j - \sum_{i=1}^{N} p_{ij}\right) + \sum_{i=1}^{N} \alpha_{ij} x_{ij} + \sum_{i=1}^{N} \beta_{ij} p_{ij}, \quad (23)$$

where all Lagrange multipliers are positive. The stationarity conditions for the Lagrangian are,

$$\frac{\partial \mathcal{L}_j}{\partial x_{ij}} = \eta_i \log\left(1 + \frac{h_{ij} p_{ij}}{x_{ij}}\right) - \frac{\eta_i h_{ij} p_{ij}}{x_{ij} + h_{ij} p_{ij}} - \mu + \alpha_{ij} = 0 \quad (24)$$

$$\frac{\partial \mathcal{L}_j}{\partial p_{ij}} = \frac{\eta_i h_{ij} x_{ij}}{x_{ij} + h_{ij} p_{ij}} - \gamma_j + \beta_{ij} = 0. \quad (25)$$

We are now in a position to state the following lemma,

Lemma 2: In the optimal solution $p_{ij} = 0$ if and only if $x_{ij} = 0$.

Proof: The result follows immediately by re-writing the (25) as,

$$p_{ij} = x_{ij} \left(\frac{\eta_i}{\gamma_j - \beta_{ij}} - \frac{1}{h_{ij}}\right), \quad (26)$$

which holds for all values of p_{ij} and x_{ij}. ■

Lemma 2 has an intuitive explanation. If no power is put in the chunk of spectrum bought from SP i, the utility term $\eta_i x_{ij} \log(1 + h_{ij} p_{ij}/x_{ij}) = 0$. If $x_{ij} \neq 0$, then user j still pays the price μx_{ij}, which clearly brings down his net utility and hence he sets $x_{ij} = 0$. Alternatively if no spectrum is bought from SP i, i.e if $x_{ij} = 0$, then automatically $p_{ij} = 0$. Henceforth we call the SPs for which $x_{ij} > 0$ and $p_{ij} > 0$ as *active* SPs.

Consider the case when $x_{ij} > 0$ and $p_{ij} > 0$, which implies that $\alpha_{ij} = 0$ and $\beta_{ij} = 0$. From (26) we obtain,

$$p_{ij} = x_{ij} \left(\frac{\eta_i}{\gamma_j} - \frac{1}{h_{ij}}\right). \quad (27)$$

Substituting for $h_{ij} p_{ij}/x_{ij}$ from (27) in (24) we obtain,

$$\frac{\gamma_j/\eta_i}{h_{ij}} = \log\left(\frac{\gamma_j/\eta_i}{h_{ij}}\right) + \frac{\mu}{\eta_i} + 1. \quad (28)$$

From (28) we again see that the effect of η_i is to modify the spectrum price μ and power level parameter γ_j to μ/η_i and γ_j/η_i respectively. We now state our second lemma,

Lemma 3: In the optimal solution only one SP is active per user almost surely.

Proof: Assume that for user j, two SPs are active and are indexed by $i = 1, 2$. Hence from (28) we obtain,

$$\frac{\gamma_j/\eta_1}{h_{1j}} - \log\left(\frac{\gamma_j/\eta_1}{h_{1j}}\right) - \frac{\mu}{\eta_1} = \frac{\gamma_j/\eta_2}{h_{2j}} - \log\left(\frac{\gamma_j/\eta_2}{h_{2j}}\right) - \frac{\mu}{\eta_2}.$$

Since h_{ij} is a continuous random variable the probability of the above event is zero. Hence only one SP is active *almost surely*. ■

Let this SP be indexed by $i^*(j)$. Then $p_{i^*(j)j} = P_j$, i.e. as user j is transmitting to only one SP he transmits with full power. Thus from (20), $i^*(j)$ is given by the solution of,

$$i^*(j) = \arg\max_i \left[\max_{x_{ij} \geq 0} \left\{\eta_i x_{ij} \log\left(1 + \frac{h_{ij} P_j}{x_{ij}}\right) - \mu x_{ij}\right\}\right].$$

To simplify notation we denote $x_{i^*(j)j}$ by x_j^*, $h_{i^*(j)j}$ by h_j^* and $\eta_{i^*(j)}$ by η_j^*. Thus the optimization problem in (20) can be re-written by considering only $i = i^*(j)$ as,

$$\mathcal{U}(x_j^*) = \max_{x_j^* \geq 0} \eta_j^* x_j^* \log\left(1 + \frac{h_j^* P_j}{x_j^*}\right) - \mu x_j^*. \quad (29)$$

The value of $\mathcal{U}(x_j^*)$ can be derived by considering (24) for $i = i^*(j)$. The result is,

$$\mathcal{U}(x_j^*) = \frac{\eta_j^* h_j^* x_j^* P_j}{x_j^* + h_j^* P_j}. \quad (30)$$

We now characterize the spectrum x_j^* of user j and his signal to noise ratio which is $\mathsf{snr}_j^* = h_j^* P_j / x_j^*$.

Lemma 4: For given channel coefficients h_j^*, power P_j and efficiency η_j^*, user j operates at a unique snr_j^* which is given by the solution of,

$$\Phi(\mathsf{snr}_j^*) = \log\left(1 + \mathsf{snr}_j^*\right) - \frac{\mathsf{snr}_j^*}{1 + \mathsf{snr}_j^*} = \frac{\mu}{\eta_j^*}. \quad (31)$$

From (31), the allocated spectrum x_j^* and spectrum price μ can also be determined.

Proof: (31) can be directly derived by writing down (24) for $i = i^*(j)$ and substituting $\mathsf{snr}_j^* = h_j^* P_j / x_j^*$. We have to show that it has a unique solution in snr_j^*. This is proved in Appendix I.

From $\Phi\left(\mathsf{snr}_j^*\right) = \Phi\left(h_j^* P_j / x_j^*\right) = \mu/\eta_j^*$, we obtain,

$$x_j^* = \frac{h_j^* P_j}{\Phi^{-1}\left(\mu/\eta_j^*\right)}. \quad (32)$$

We have proved in Appendix I that $\Phi(\cdot)$ is an increasing and one-to-one function; hence $\Phi^{-1}(\cdot)$ is also increasing and one-to-one. Thus there is a unique allocated spectrum x_j^*.

The spectrum price is given by solving the total spectrum constraint. We note that (3) and (5) can be jointly written as $\sum_{j=1}^{L} x_j^* \leq C$ or from (32) as,

$$\sum_{j=1}^{L} \frac{h_j^* P_j}{\Phi^{-1}\left(\mu/\eta_j^*\right)} \leq C. \quad (33)$$

Let $\tilde{\mu}$ be the solution when equality is assumed in (33). Then the optimal spectrum price is given by $\mu = \min\{\tilde{\mu}, 0\}$. When $\tilde{\mu} < 0$, it implies that constraints (3) and (5) are not tight and the entire spectrum C is not being utilized by the users. ■

The following direct observations can be made,

Observation 1: As $\Phi(\cdot)$ is an increasing function, from (31) we see that an increase in μ or decrease in η_j^* causes user j to operate at a higher snr_j^*.

Observation 2: As $\Phi^{-1}(\cdot)$ is an increasing function, from (32) we see that an increase in μ or decrease in η_j^* causes a lesser spectrum to be allocated to user j.

Thus as spectrum price goes up (possibly because of external changes like another user joining the system etc.) or the SP efficiency reduces, the user obtains less spectrum and to make up for this transmits at a higher spectrally efficient modulation scheme. Recall that spectral efficiency is $\log(1+\mathsf{snr})$.

1) $\eta_j^* = 1$ for all j: For this case we can simplify the solutions. From (32), we obtain,

$$\Phi^{-1}(\mu) = \frac{h_1^* P_1}{x_1^*} = \cdots = \frac{h_L^* P_L}{x_L^*} = \frac{\sum_{k=1}^{L} h_k^* P_k}{C}. \tag{34}$$

Thus x_j^* is given by,

$$x_j^* = \frac{h_j^* P_j}{\sum_{k=1}^{L} h_k^* P_k} C. \tag{35}$$

Thus we see that the allocated spectrum is directly proportional to the received signal power. Substituting for x_j^* in (30), we obtain,

$$\mathcal{U}(x_j^*) = \frac{h_j^* P_j}{1 + \sum_{k=1}^{L} h_k^* P_k / C}. \tag{36}$$

It can be shown that (36) is an increasing function of h_j^*. Thus for $\eta_j^* = 1$, the active SP for user j is the one which has the best channel to user j, i.e. $i^*(j) = \arg\max_i h_{ij}$.

We now see how change in users power P_j or channel h_j^* affects price μ and x_j^*.

Lemma 5: With increase in available power the spectrum price increases, an user obtains more spectrum and derives a higher utility.

Proof: Consider equality in (33) and focus on user 1. Without loss in generality we assume that P_1 increases to $P_1' > P_1$ and P_2 to P_L and h_1^* to h_L^* stay the same. Now μ can either increase, decrease or stay the same. If μ decreases, so does the term $\Phi^{-1}(\mu/\eta_i)$ and each term in the LHS of (33) increases. The first term has a further increase as $P_1' > P_1$. Thus equality can't be maintained as RHS of (33) is still C. If μ stays the same, the second to L^{th} terms in the LHS of (33) stays the same but the first term still increases as $P_1' > P_1$. Hence μ must increase to $\mu' > \mu$. Thus the allocated spectrum to users 2 to L decreases as per (32). Note that this in a manifestation of Observation 2. Since allocated spectrum for all other users decrease, the allocated spectrum for user 1, i.e. x_1^* must increase to satisfy (33).

To prove the second part of the lemma, consider (30). Let $x_j^* = x_a$ correspond to $P_j = P_a$ and $x_j^* = x_b$ to $P_j = P_b$, where $P_b > P_a$ and hence $x_b > x_a$. We can easily show that,

$$\mathcal{U}(x_b) - \mathcal{U}(x_a) = \eta_j^* h_j^* \frac{(P_b - P_a) x_b x_a + h_j^*(x_b - x_a) P_b P_a}{(x_a + h_j^* P_a)(x_b + h_j^* P_b)},$$

which is positive. ■

Note that as P_j changes from P_a to P_b so does the spectrum price μ, but the Lemma still holds as the expression for the optimum user utility in (30) is independent of μ.

Lemma 6: As channel condition to the active SP becomes better the spectrum price increases, an user obtains more spectrum and derives a higher utility.

Proof: The proof is exactly similar to proof of Lemma 5, with h_j^* now being the variable instead of P_j. ■

We now consider how the dynamics of the system changes as more users or SPs are added to it.

Lemma 7: As more users are added to the system, the spectrum price either stays the same or increases.

Proof: Assume that the system is in equilibrium and one more user joins in the system. If he is not admitted into the system (the new optimal solution does not allocate any spectrum to him), the status quo is maintained and the old price still holds. However if he is allocated some chunk of spectrum, then every other user's allocated spectrum decreases (as the channel conditions and power of these users are still the same). But as noted in Observation 2, a decrease in x_{ij}, increases the value of spectrum price μ.■

We can interpret the result as follows: As more users are added into the system, there is higher demand for spectrum. As a result of increases demand, the price of spectrum also increases and as other resources of the user, like power and channel conditions are same, he is forced to obtain a lesser chunk of spectrum.

Lemma 8: As one more SP is added to the system the spectrum price either stays the same or increases.

Proof: Assume that the system is in equilibrium and a new SP (SP $N+1$) joins in the system. If it offers no better channel to any of the users than their existing ones, i.e. if $h_j^* > h_{(N+1)j}$ for all j, then no user engages itself to the SP and the optimal solution (spectrum price, spectrum allocated etc) is the same as before.

On the other hand if for some user j, the new SP provides a better channel coefficient, i.e $h_j^* < h_{(N+1)j}$, then user j engages itself to SP $N+1$ and adjusts its engaged SP index to $i^*(j) = N+1$ and channel coefficient to $h_j^* = h_{(N+1)j}$. Thus user j's channel condition to his active SP has improved and as per Lemma 6, the price goes up. ■

The result might seem surprising as we often associate more service providers with lesser prices. However that happens when *supply* of a commodity (like total available spectrum C) increases with more service providers but in our case C is independent of the number of service providers. The effect of more SPs shows up indirectly as they provide users with a bigger set of channel coefficients to choose from.

III. Various Comments

In this section we present some comments about our problem formulation and solution.

A. Dynamic Allocation

In this work, we have assumed that the system parameters like h_{ij}, N and L stay constant for the duration in which the optimization is carried out and for the subsequent transmission time. Whenever there is a significant change in these parameters the optimization can be re-done. This demonstrates the dynamic nature of our spectrum allocation. However to avoid too rapid changes in spectrum allocation which could arise if the short term fading is considered in the variation of h_{ij}, the optimization in Section II-A could be done for the long term channel attenuation factors like path loss and shadowing under the assumption that short term fading can be combated at the receiver by signal processing techniques like diversity combiners.

B. Orthogonal Spectrum Allocation

Motivated by 802.22 systems we have considered zero interference. In our system users are allocated non-overlapping chunks of spectrum making them are orthogonal in frequency domain. As number of users increase, the system will reach a critical number of users L^* such that the optimization problem (1) to (7) is feasible for $L = L^*$ but not for $L = L^* + 1$. Physically it means that user $L^* + 1$ will be blocked from accessing spectrum and the performance of the original L^* users will stay the same. Such blocking behavior is common to other orthogonal access systems like telephony where there are finite number of switches and each call is given a unique switch.

In contrast there are non-orthogonal systems where the available spectrum is shared equally by all users in the system, like Bluetooth and 802.11 sharing the UNII band. There is no concept of blocking and with addition of additional devices the performance of all systems go down uniformly [10].

IV. Conclusion

In this work we mathematically analyze a two tier spectrum allocation system consisting multiple SPs providing spectrum to users. The spectrum allocation is facilitated by a SCH which leads to simple SP coordination and distributed spectrum allocation at the users. We model the above system from an user welfare maximization framework. We show that in the optimal policy each user obtains spectrum only from one service provider. We introduce the notion of spectrum price and show that this facilitates distributed spectrum allocation for the users. For utility functions that are linear with transmission rates we completely characterize the behavior of the spectrum allocation solution. We believe that the model presented here forms the starting step to analyze more complex networks where apart from user welfare maximization, service providers also want to maximize their own revenues.

Appendix I
Proof of Uniqueness in Lemma 4

Prove that the equation,

$$\Phi(x) = \log(1+x) - \frac{x}{1+x} = \mu, \tag{37}$$

where $x > 0$ has a unique solution which is increasing in μ.

A. Solution

Taking derivatives we obtain,

$$\Phi'(x) = \frac{x}{(1+x)^2} > 0 \text{ for } x > 0. \tag{38}$$

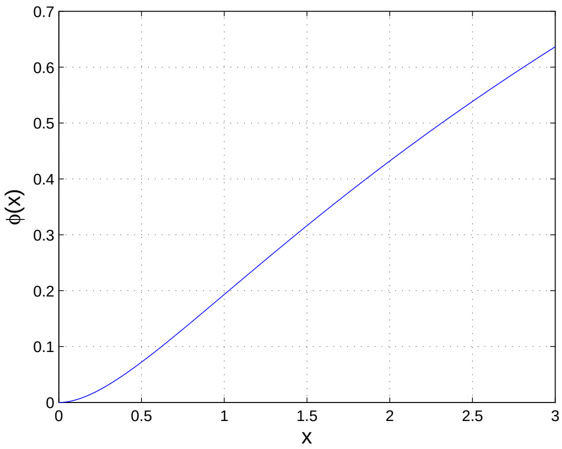

Fig. 2. $\Phi(x)$ vs x

Thus $\Phi(x)$ is a strictly increasing function. Thus it is also one-to-one, i.e. a value of μ yields a unique value of x. By taking the second derivative we obtain,

$$\Phi''(x) = \frac{1-x}{(1+x)^3}, \tag{39}$$

which shows that there is a point of inflection at $x = 1$. A sample $\Phi(x)$ is plotted in Figure 2

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