Complementary Set Matrices Satisfying a Column Correlation Constraint

Di Wu and Predrag Spasojević

WINLAB, Rutgers University

671 Route 1 South, North Brunswick, NJ 08902

{diwu,spasojev}@winlab.rutgers.edu

Abstract

Motivated by the problem of reducing the peak to average power ratio (PAPR) of transmitted signals, we consider a design of complementary set matrices whose column sequences satisfy a correlation constraint. The design algorithm recursively builds a collection of 2^{t+1} mutually orthogonal (MO) complementary set matrices starting from a companion pair of sequences. We relate correlation properties of column sequences to that of the companion pair and illustrate how to select an appropriate companion pair to ensure that a given column correlation constraint is satisfied. For $t = 0$, companion pair properties directly determine matrix column correlation properties. For $t \geq 1$, reducing correlation merits of the companion pair may lead to improved column correlation properties. However, further decrease of the maximum out-off-phase aperiodic autocorrelation of column sequences is not possible once the companion pair correlation merit is less than a threshold determined by t. We also reveal a design of the companion pair which leads to complementary set matrices with Golay column sequences. Exhaustive search for companion pairs satisfying a column correlation constraint is infeasible for medium and long sequences. We instead search for two shorter length sequences by minimizing a cost function in terms of their autocorrelation and crosscorrelation merits. Furthermore, an improved cost function which helps in reducing the maximum out-off-phase column correlation is derived based on the properties of the companion pair. By exploiting the well-known Welch bound, sufficient conditions for the existence of companion pairs which satisfy a set of column correlation constraints are also given.

Index Terms

Complementary sets, Golay sequences, peak to average power ratio (PAPR), Welch bound

* This work has been supported in part by the NSF Grant ANI 0338805

I. Introduction

Complementary sequence sets have been introduced by Golay [1], [2], as a pair of binary sequences with the property that the sum of their aperiodic autocorrelation functions (ACF) is zero everywhere except at zero shift. Tseng and Liu [3] generalized these ideas to sets of binary sequences of size larger than two. Sivaswamy [4] and Frank [5] investigated the multiphase (polyphase) complementary sequence sets with constant amplitude sequence elements. Gavish and Lemple considered ternary complementary pairs over the alphabet $\{1, 0, -1\}$ [6]. The synthesis of multilevel complementary sequences is described in [7]. These generalizations of a binary alphabet lead to new construction methods for complementary sets having a larger family of lengths and cardinalities. However, all these studies focus either on the set complementarity or on the design of orthogonal families of complementary sets. Correlation properties of column sequences of the complementary set matrix (i.e., the matrix whose row sequences form a complementary set) have not been considered.

In [8]–[10], a technique for the multicarrier direct-sequence code-division multiple access (MC-DS-CDMA) system [11], [12] that employs complementary sets as spreading sequences has been investigated. Each user assigns different sequences from a complementary set to his subcarriers. By assigning mutually orthogonal (MO) complementary sets to different users, both multiple access interference and multipath interference can be significantly suppressed. Similar to conventional multicarrier systems, one of the major impediments to deploying such systems is high peak-to-average power ratio (PAPR). We have stressed in [13] that correlation properties of column sequences of complementary set matrices play an important role in the reduction of PAPR. In this work, we search for ways of constructing complementary set matrices whose column sequences satisfy a correlation constraint.

For orthogonal frequency-division multiplexing (OFDM) signaling, Tellambura [14] derived a general upper bound on the signal peak envelope power (PEP) in terms of the aperiodic ACF of the sequence whose elements are assigned across all signal carriers. He has shown that sequences with small aperiodic autocorrelation values can reduce the PAPR of the OFDM signal. By generalizing earlier work of Boyd [15], Popović [16] has demonstrated that PAPR corresponding to any binary Golay sequence (i.e., a sequence having a Golay complementary pair) is at most two. This has motivated Davis and Jedwab to explicitly determine a large class of Golay sequences as a solution to the signal envelope problem [17]. Here, we consider sequence sets which are characterized by both their complementarity and a desired column correlation constraint.

We describe a construction algorithm for the design of 2^{t+1} MO complementary set matrices of size $2^t m$ by

2^{t+p+1} in Section III, where t and p can be any non-negative integer, and m is an even number. The construction process is based on a set of sequence/matrix operations, starting from a two column matrix formed by a companion sequence pair. These operations preserve the alphabet (up to the sign) of the companion pair. In Section IV, we illustrate how, by selecting an appropriate companion pair we can ensure that column sequences of the constructed complementary set matrix satisfy a correlation constraint. For $t = 0$, companion pair properties directly determine matrix column correlation properties. For $t \geq 1$, reducing correlation merits of the companion pair may lead to improved column correlation properties. However, further decrease of the maximum out-off-phase aperiodic autocorrelation of column sequences is not possible once the correlation of the companion pair is less than a threshold determined by t. We also present a method for constructing the companion pair which leads to the complementary set matrix with Golay column sequences.

In Section V, an algorithm for searching for companion pairs over length m sequences of a desired alphabet is described. However, exhaustive search is infeasible for medium and long sequences. We instead suggest finding companion pairs with a small, if not minimum, column correlation constraint. In Section VI, by exploiting properties of the companion pair, we convert the problem into a search for two sequences of length $m/2$ with low autocorrelation and crosscorrelation merits, a long standing problem in literature (e.g. see [18]–[21]). We further derive an improved cost function and show how it leads to reduced achievable maximum out-off-phase column correlation constraint. Sufficient conditions for the existence of companion pairs which satisfy various column correlation constraints are also derived. We conclude in Section VII.

II. Definitions and Preliminaries

Throughout this paper, sequences are denoted by boldface lowercase letters (e.g., $\mathbf{x}$), their elements by corresponding lowercase letters with subscripts (x_0), boldface uppercase letters denote matrices ($\mathbf{X}$), and calligraphic letters denote either sets of numbers or sets of sequences ($\mathcal{X}$).

A. Correlation functions

Let $\mathbf{a} = (a_0, a_1, ..., a_{n-1})$ denote a sequence of length n with $a_i \in \mathcal{C}$, $0 \leq i \leq n-1$, where $\mathcal{C}$ is the set of complex numbers. The aperiodic and periodic ACFs of $\mathbf{a}$ are

$$A_{\mathbf{a}}(l) = \sum_{i=0}^{n-1-l} a_i a_{i+l}^*, \quad 0 \leq l \leq n-1 \tag{1}$$

$$P_{\mathbf{a}}(l) = \sum_{i=0}^{n-1} a_i a_{i \oplus_n l}^*, \quad 0 \leq l \leq n-1, \tag{2}$$

TABLE I

CORRELATION MERITS

$\lambda_{\mathbf{a}}^{A}$	=	$\max_l\{\lvert A_{\mathbf{a}}(l)\rvert, 1 \leq l \leq n-1\}$	$S_{\mathbf{a}}^{A}$	=	$\sum_{l=1}^{n-1} \lvert A_{\mathbf{a}}(l)\rvert$
$\lambda_{\mathbf{a}}^{P}$	=	$\max_l\{\lvert P_{\mathbf{a}}(l)\rvert, 1 \leq l \leq n-1\}$	$S_{\mathbf{a}}^{P}$	=	$\sum_{l=1}^{n-1} \lvert P_{\mathbf{a}}(l)\rvert$
$\lambda_{\mathbf{a},\mathbf{b}}^{A}$	=	$\max_l\{\lvert A_{\mathbf{a},\mathbf{b}}(l)\rvert, \lvert l \rvert \leq n-1\}$	$S_{\mathbf{a},\mathbf{b}}^{A}$	=	$\sum_{l=1-n}^{n-1} \lvert A_{\mathbf{a},\mathbf{b}}(l)\rvert$
$\lambda_{\mathbf{a},\mathbf{b}}^{P}$	=	$\max_l\{\lvert P_{\mathbf{a},\mathbf{b}}(l)\rvert, 0 \leq l \leq n-1\}$	$S_{\mathbf{a},\mathbf{b}}^{P}$	=	$\sum_{l=0}^{n-1} \lvert P_{\mathbf{a},\mathbf{b}}(l)\rvert$

where a^* denotes the complex conjugate of a, and $\oplus_n$ denotes modulo-n addition. It follows that

$$P_{\mathbf{a}}(l) = A_{\mathbf{a}}(l) + A_{\mathbf{a}}(n-l), \quad 0 \leq l \leq n-1. \tag{3}$$

Let $\mathbf{b} = (b_0, b_1, ..., b_{n-1})$, where $b_i \in \mathcal{C}$, $0 \leq i \leq n-1$. The aperiodic and periodic crosscorrelation functions of $\mathbf{a}$ and $\mathbf{b}$ are defined, respectively, as,

$$A_{\mathbf{a},\mathbf{b}}(l) = \begin{cases} \sum_{i=0}^{n-1-l} a_i b_{i+l}^*, & 0 \leq l \leq n-1 \\ \sum_{i=0}^{n-1+l} a_{i-l} b_i^*, & 1-n \leq l < 0 \\ 0, & |l| \geq n \end{cases} \tag{4}$$

$$P_{\mathbf{a},\mathbf{b}}(l) = \sum_{i=0}^{n-1} a_i b_{i \oplus_n l}^*, \quad 0 \leq l \leq n-1 \ . \tag{5}$$

Table I lists the correlation function parameters which are commonly used to judge the merits of a sequence design, and are termed *correlation merits*. $\lambda_{\mathbf{a}}^{A}$, $\lambda_{\mathbf{a}}^{P}$, $S_{\mathbf{a}}^{A}$, and $S_{\mathbf{a}}^{P}$ are autocorrelation merits, and $\lambda_{\mathbf{a},\mathbf{b}}^{A}$, $\lambda_{\mathbf{a},\mathbf{b}}^{P}$, $S_{\mathbf{a},\mathbf{b}}^{A}$, and $S_{\mathbf{a},\mathbf{b}}^{P}$ are common crosscorrelation merits. For example, it is well known that binary m-sequences satisfy $\lambda_{\mathbf{a}}^{P} = 1$, and the sequences with $\lambda_{\mathbf{a}}^{A} \leq 1$ are called Barker sequences. Furthermore, a small value of $S_{\mathbf{a}}^{A}$ can significantly reduce the PAPR of OFDM signals, if the elements of $\mathbf{a}$ are assigned across all carriers [13], [14].

B. Complementary sets and column correlation constraints

A set of m sequences $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_m\}$, each of length n, is called a complementary set if

$$\sum_{i=1}^{m} A_{\mathbf{a}_i}(l) = 0, \quad 1 \leq l \leq n-1 \ . \tag{6}$$

When $m = 2$, the binary sequence set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is called a Golay complementary pair and $\mathbf{a}_i$, $i = 1, 2$, are Golay sequences. They are known to exist for all lengths $n = 2^{\alpha} 10^{\beta} 26^{\gamma}$, where $\alpha, \beta, \gamma \geq 0$ [22], [23]. Complementary sets $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m\}$ and $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_m\}$ are mates if

$$\sum_{i=1}^{m} A_{\mathbf{a}_i,\mathbf{b}_i}(l) = 0, \tag{7}$$

for every l. The MO complementary set is a collection of complementary sets in which any two are mates. Let $\mathbf{M}_{m,n}^{k}$ denote the MO complementary set consisting of k complementary sets each having m complementary sequences of length n. For binary sequences, k cannot exceed m [23], that is, the maximum number of mutually orthogonal complementary sets is equal to the number of complementary sequences in a set. Hence, $\mathbf{M}_{m,n}^{m}$ is called *a complete complementary code of order* m [24].

A complementary set can be represented using a *complementary set matrix*

$$\mathbf{C} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} = \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_{n-1} \end{bmatrix}^T = \begin{bmatrix} c_{1,0} & c_{1,1} & \cdots & c_{1,n-1} \\ c_{2,0} & c_{2,1} & \cdots & c_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m,0} & c_{m,1} & \cdots & c_{m,n-1} \end{bmatrix}_{m\times n} \tag{8}$$

where T denotes the matrix transpose. *Row sequences*, $\mathbf{r}_i = (c_{i,0}, c_{i,1}, c_{i,2}, \cdots, c_{i,n-1})$, $1 \leq i \leq m$, are complementary sequences, that is, $\sum_{i=1}^{m} A_{\mathbf{r}_i}(l) = 0$, $1 \leq l \leq n-1$. The main focus of this paper are properties of *column sequences*, $\mathbf{c}_j = (c_{1,j}, c_{2,j}, c_{3,j}, \cdots, c_{m,j})$, $0 \leq j \leq n-1$.

A set of k mutually orthogonal complementary sets, $\{\mathbf{C}_1, \mathbf{C}_2, ...\mathbf{C}_k\}$, form a MO complementary set matrix

$$\mathbf{M}_{m,n}^{k} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 & \cdots & \mathbf{C}_k \end{bmatrix}_{m\times kn} \tag{9}$$

and its column sequences are denoted as $\mathbf{u}_i, 0 \leq i \leq kn-1$. An upper bound on an autocorrelation merit of column sequences is termed *the column correlation constraint*. If there exists at least one MO complementary set matrix satisfying a given column correlation constraint, then this constraint is called *an achievable column correlation constraint*. For example, let

$$\lambda_{\mathbf{u}}^{A} = \max\left\{\lambda_{\mathbf{u}_i}^{A}, 0 \leq i \leq kn-1\right\}, \tag{10}$$

if $\lambda_{\mathbf{u}}^{A} \leq \lambda^{A}$, then $\mathbf{M}_{m,n}^{k}$ is called a MO complementary set matrix satisfying a column correlation constraint λ^{A}, and λ^{A} is an achievable column correlation constraint. We also consider column sequences which are Golay sequences, i.e., Golay column sequences.

C. Companion pair

Let $\mathbf{a}$ be a sequence of length m, where m is an even number. We define a sequence $\mathbf{b}$ as a companion of $\mathbf{a}$ if

$$\mathbf{C} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^T \tag{11}$$

is a complementary set matrix which consists of $m/2$ complementary pairs. $\mathbf{C}$ is called *a companion matrix* and $(\mathbf{a}, \mathbf{b})$ is called *a companion pair*.

TABLE II

SEQUENCE OPERATIONS

$\overleftarrow{\mathbf{a}}$	=	$(a_{n-1}, a_{n-2}, ..., a_1, a_0)$
$-\mathbf{a}$	=	$(-a_0, -a_1, ..., -a_{n-1})$
$\mathbf{a}^*$	=	$(a_0^*, a_1^*, ..., a_{n-1}^*)$
$\mathbf{ab}$	=	$(a_0, a_1, ..., a_{n-1}, b_0, b_1, ..., b_{n-1})$
$\mathbf{a}\otimes\mathbf{b}$	=	$(a_0, b_0, a_1, b_1, ..., a_{n-1}, b_{n-1})$
$\mathbf{a}\cdot\mathbf{b}$	=	$a_0b_0 + a_1b_1 + ... + a_{n-1}b_{n-1}$
$f_i(\mathbf{a})$	=	$(a_1, -a_0, a_3, -a_2, ..., a_{n-1}, -a_{n-2})$
$f_c(\mathbf{a})$	=	$(a_{\frac{n}{2}}, a_{\frac{n}{2}+1}, ..., a_{n-1}, -a_0, -a_1, ..., -a_{\frac{n}{2}-1})$

D. Operations and extensions

1) Sequence and matrix operations: In Table II, we list the following sequence operations: reversal, negation, complex conjugation, concatenation, interleaving, and inner product. Furthermore, we introduce two sequence reshaping functions $f_i(\cdot)$ and $f_c(\cdot)$ defined on sequences of even length.

Let $\mathbf{C} = [c_{i,j}]$ and $\mathbf{D}$ be two matrices of equal dimensions, then $\mathbf{C}^* = [c_{i,j}^*]$ and $-\mathbf{C} = [-c_{i,j}]$. $\mathbf{C} \otimes \mathbf{D}$ is the matrix whose ith row sequence is obtained by interleaving ith row sequences of $\mathbf{C}$ and $\mathbf{D}$. $\mathbf{CD}$ denotes the matrix whose ith row sequence is the concatenation of ith row sequences of $\mathbf{C}$ and $\mathbf{D}$.

Let us also define a sequence set $\mathcal{R}_{\mathbf{c}}^{(v)}$, which is a collection of row sequences of matrix $\mathbf{R}_{\mathbf{c}}^{(v)}$ recursively constructed from a sequence $\mathbf{c}$, as follows,

$$\mathbf{R}_{\mathbf{c}}^{(v)} = \begin{bmatrix} \mathbf{R}_{\mathbf{c}}^{(v-1)}\mathbf{R}_{\mathbf{c}}^{(v-1)} \\ \mathbf{R}_{\mathbf{c}}^{(v-1)}(-\mathbf{R}_{\mathbf{c}}^{(v-1)}) \end{bmatrix}, \quad v = 1, 2, 3... \tag{12}$$

where

$$\mathbf{R}_{\mathbf{c}}^{(0)} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}_{2\times m} . \tag{13}$$

Let $\mathcal{R}_{\mathbf{c},\mathbf{d}}^{(v)} = \mathcal{R}_{\mathbf{c}}^{(v)} \cup \mathcal{R}_{\mathbf{d}}^{(v)}$ which consists of 2^{v+2} sequences of length $2^v m$. For example, $\mathcal{R}_{\mathbf{c}}^{(0)} = \{\mathbf{c}, -\mathbf{c}\}$, $\mathcal{R}_{\mathbf{d}}^{(0)} = \{\mathbf{d}, -\mathbf{d}\}$, and $\mathcal{R}_{\mathbf{c},\mathbf{d}}^{(0)} = \{\mathbf{c}, \mathbf{d}, -\mathbf{c}, -\mathbf{d}\}$.

2) Complementary set matrix extension operations: We describe two operations for extending complementary set matrices, namely, *length-extension* and *size-extension*.

Lemma 2.1 [23]: Let $\{\mathbf{a}, \mathbf{b}\}$ be a complementary pair, then $\{\overleftarrow{\mathbf{b}}^*, -\overleftarrow{\mathbf{a}}^*\}$ is its mate, and both $\{\mathbf{a}\overleftarrow{\mathbf{b}}^*, \mathbf{b}(-\overleftarrow{\mathbf{a}}^*)\}$ and $\{\mathbf{a} \otimes \overleftarrow{\mathbf{b}}^*, \mathbf{b} \otimes (-\overleftarrow{\mathbf{a}}^*)\}$ are complementary pairs.

Proof: See Appendix A. ■

Lemma 2.2: Let $\{\mathbf{a}_1, \mathbf{b}_1\}$, $\{\mathbf{a}_2, \mathbf{b}_2\}$, ..., $\{\mathbf{a}_m, \mathbf{b}_m\}$ be m complementary pairs of length n. Then, $\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, ..., \mathbf{a}_m, \mathbf{b}_m\}$ is a complementary set of $2m$ complementary sequences.

Proof: $\sum_{i=1}^{m} (A_{\mathbf{a}_i}(l) + A_{\mathbf{b}_i}(l)) = 0, \;\; 1 \leq l \leq n-1.$ ■

Lemmas 2.1-2.2 imply that, if a complementary set consists of $m/2$ complementary pairs, the sequence length can be recursively doubled as follows. Let

$$\mathbf{C}^{(p)} = \begin{bmatrix} \mathbf{r}_1^{(p)} \\ \mathbf{r}_2^{(p)} \\ \vdots \\ \mathbf{r}_{m-1}^{(p)} \\ \mathbf{r}_m^{(p)} \end{bmatrix}_{m \times n^{(p)}} \quad \text{and} \quad \mathbf{D}^{(p)} = \begin{bmatrix} \overleftarrow{\mathbf{r}}_2^{(p)} \\ -\overleftarrow{\mathbf{r}}_1^{(p)} \\ \vdots \\ \overleftarrow{\mathbf{r}}_m^{(p)} \\ -\overleftarrow{\mathbf{r}}_{m-1}^{(p)} \end{bmatrix}^*_{m \times n^{(p)}} \tag{14}$$

be, respectively, an m by $n^{(p)}$ complementary set matrix and its mate, where m is an even number, and $\{\mathbf{r}_1^{(p)}, \mathbf{r}_2^{(p)}\}, \{\mathbf{r}_3^{(p)}, \mathbf{r}_4^{(p)}\}, ...\{\mathbf{r}_{m-1}^{(p)}, \mathbf{r}_m^{(p)}\}$ are assumed to be complementary pairs. A complementary set matrix $\mathbf{C}^{(p+1)}$ of dimension m by $n^{(p+1)} = 2n^{(p)}$ can be constructed recursively as either,

$$\mathbf{C}^{(p+1)} = \mathbf{C}^{(p)} \; \mathbf{D}^{(p)} \tag{15}$$

$$\text{or} \qquad \mathbf{C}^{(p+1)} = \mathbf{C}^{(p)} \otimes \mathbf{D}^{(p)}. \tag{16}$$

We term (15) and (16) *length-extension operations*.

Lemma 2.3 [3]: A MO complementary set matrix $\mathbf{M}^{2k}_{2m,2n}$ can be constructed recursively as either,

$$\mathbf{M}^{2k}_{2m,2n} = \begin{bmatrix} \mathbf{M}^k_{m,n}\mathbf{M}^k_{m,n}, (-\mathbf{M}^k_{m,n})\mathbf{M}^k_{m,n} \\ (-\mathbf{M}^k_{m,n})\mathbf{M}^k_{m,n}, \mathbf{M}^k_{m,n}\mathbf{M}^k_{m,n} \end{bmatrix} \tag{17}$$

or

$$\mathbf{M}^{2k}_{2m,2n} = \begin{bmatrix} \mathbf{M}^k_{m,n}\otimes\mathbf{M}^k_{m,n}, (-\mathbf{M}^k_{m,n})\otimes\mathbf{M}^k_{m,n} \\ (-\mathbf{M}^k_{m,n})\otimes\mathbf{M}^k_{m,n}, \mathbf{M}^k_{m,n}\otimes\mathbf{M}^k_{m,n} \end{bmatrix} \tag{18}$$

Proof: Refer to the proof of *Theorem 12-13* in [3]. ■

We term (17) and (18) *size-extension operations*.

III. Construction of complementary set matrices from a companion pair

A. Recursive construction

Let $\mathcal{X}(m)$ denote a sequence set which consists of all length m sequences whose elements are from the alphabet $\mathcal{X}$. We summarize a recursive construction of a MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ with elements from $\mathcal{X}$, where $n^{(p)} = 2^{p+1}$, m is an even number, and $t, p = 0, 1, 2, ...$

Step 1: The construction starts from a companion pair $\mathbf{c}_0$ and $\mathbf{c}_1$ which are in $\mathcal{X}(m)$. They form an m by 2 companion matrix

$$\mathbf{C}^{(0)} = \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \end{bmatrix}^T . \tag{19}$$

Step 2: By employing the length-extension operation p times, we extend $\mathbf{C}^{(0)}$ to an m by $n^{(p)} = 2^{p+1}$ complementary set matrix $\mathbf{C}^{(p)}$. $\mathbf{C}^{(p)}$ and its mate $\mathbf{D}^{(p)}$ constructed from Eq. (14) form a MO complementary set matrix

$$\mathbf{M}^2_{m,n^{(p)}} = \begin{bmatrix} \mathbf{C}^{(p)} & \mathbf{D}^{(p)} \end{bmatrix}_{m \times 2^{p+2}} . \tag{20}$$

Step 3: Starting with $\mathbf{M}^2_{m,n^{(p)}}$, we can construct the MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ by repeating the size-extension operation t times, where $p, t = 0, 1, 2, ...$

In this paper, we will alternately use either "the constructed MO complementary set matrix" or, simply, $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ when referring to the above constructed MO complementary set matrix.

B. Companion pair design and properties

Proposition 3.1: Let us arrange the elements of $\mathbf{c}_0 = (c_{1,0}, c_{2,0}, ...c_{m,0})$ into $m/2$ arbitrary pairs, e.g., $(c_{x,0}, c_{y,0})$. Then, its companion sequence $\mathbf{c}_1 = (c_{1,1}, c_{2,1}, ...c_{m,1})$ can be constructed as either

$$c_{x,1} = c^*_{y,0}, \quad c_{y,1} = -c^*_{x,0} \tag{21}$$

$$\text{or} \qquad c_{x,1} = -c^*_{y,0}, \quad c_{y,1} = c^*_{x,0} \tag{22}$$

Proof: Let us assume $\mathbf{c}_1$ is constructed using (21). In this case,

$$\mathbf{C}^{(0)} = \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \end{bmatrix}^T = \begin{bmatrix} \dots & c_{x,0} & \dots & c_{y,0} & \dots \\ \dots & c^*_{y,0} & \dots & -c^*_{x,0} & \dots \end{bmatrix}^T . \tag{23}$$

Here, $\mathbf{r}^{(0)}_x = (c_{x,0}, c^*_{y,0})$ and $\mathbf{r}^{(0)}_y = (c_{y,0}, -c^*_{x,0})$, respectively, the xth and the yth row sequence of $\mathbf{C}^{(0)}$ form a complementary pair, since

$$A_{\mathbf{r}^{(0)}_x}(l) + A_{\mathbf{r}^{(0)}_y}(l) = 0, \ 1 \le l < 2. \tag{24}$$

Based on *Lemma 2.2*, $\mathbf{C}^{(0)}$ is a complementary set matrix consisting of $m/2$ complementary pairs. Hence, $\mathbf{c}_0$ and $\mathbf{c}_1$ form a companion pair. ■

Example 3.1: $f_i^*(\mathbf{c}_0)$ is a companion of $\mathbf{c}_0$, since

$$\mathbf{C}^{(0)} = \begin{bmatrix} \mathbf{c}_0 \\ f_i^*(\mathbf{c}_0) \end{bmatrix}^T = \begin{bmatrix} c_{1,0} & c_{2,0} & c_{3,0} & c_{4,0} & \cdots & c_{m-1,0} & c_{m,0} \\ c_{2,0}^* & -c_{1,0}^* & c_{4,0}^* & -c_{3,0}^* & \cdots & c_{m,0}^* & -c_{m-1,0}^* \end{bmatrix}^T \tag{25}$$

is a companion matrix. It can be verified that $f_c^*(\mathbf{c}_0)$ is also a companion of $\mathbf{c}_0$.

The companion pair has the following properties,

Property 1 (Commutative property): If $\mathbf{c}_0$ is a companion of $\mathbf{c}_1$, then $\mathbf{c}_1$ is also a companion of $\mathbf{c}_0$.

Proof: In Eq. (11), $\mathbf{C}$ is still a companion matrix when the column sequences $\mathbf{a}$ and $\mathbf{b}$ are switched. ■

Property 2 (Inner product property): If $\mathbf{c}_0$ and $\mathbf{c}_1$ form a companion pair, then $\mathbf{c}_0 \cdot \mathbf{c}_1 = 0$.

Proof: Based on Eqs. (21) and (22), $\sum_{i=1}^{m} c_{i,0}c_{i,1} = 0$. ■

Corollary 3.1: Binary sequences $\mathbf{c}_0$ and $\mathbf{c}_1$ form a companion pair, if and only if $\mathbf{c}_0 \cdot \mathbf{c}_1 = 0$.

Proof: From Property 2, $\mathbf{c}_0 \cdot \mathbf{c}_1 = 0$ is for any companion pair. Furthermore, if $\mathbf{c}_0$ and $\mathbf{c}_1$ are binary sequences such that $\mathbf{c}_0 \cdot \mathbf{c}_1 = 0$, there must exist $m/2$ pairs of (x, y) satisfying $c_{x,0}c_{x,1} + c_{y,0}c_{y,1} = 0$, where $1 \leq x \leq m$ and $1 \leq y \leq m$. Hence, row sequences of $\mathbf{C}^{(\mathbf{0})}$ can be arranged into $m/2$ pairs, where each pair $\mathbf{r}_x^{(0)} = (c_{x,0}, c_{x,1})$ and $\mathbf{r}_y^{(0)} = (c_{y,0}, c_{y,1})$ satisfies $A_{\mathbf{r}_x^{(0)}}(l) + A_{\mathbf{r}_y^{(0)}}(l) = 0$, $1 \leq l < 2$. Consequently, $\mathbf{C}^{(0)}$ is a companion matrix. ■

C. Column sequence properties

Lemma 3.1: All column sequences of the constructed complementary set matrix $\mathbf{M}^2_{m,n^{(p)}} = \begin{bmatrix}\mathbf{C}^{(p)} & \mathbf{D}^{(p)}\end{bmatrix}$ are in $\mathcal{R}^{(0)}_{\mathbf{c}_0,\mathbf{c}_1} = \{\pm\mathbf{c}_0, \pm\mathbf{c}_1\}$, where $(\mathbf{c}_0, \mathbf{c}_1)$ is the companion pair for the construction.

Proof: Let

$$\mathbf{C}^{(p)} = \begin{bmatrix} \mathbf{c}_0^{(p)} \\ \mathbf{c}_1^{(p)} \\ \vdots \\ \mathbf{c}_{n^{(p)}-1}^{(p)} \end{bmatrix}^T . \tag{26}$$

For $\mathbf{C}^{(p+1)}$ constructed by the length-extension (15), we have,

$$\mathbf{c}_i^{(p+1)} = \begin{cases} \mathbf{c}_i^{(p)} & 0 \leq i \leq n^{(p)} - 1 \\ -\mathbf{c}_{i-n^{(p)}}^{(p)} & n^{(p)} \leq i \leq \frac{3}{2}n^{(p)} - 1 \\ \mathbf{c}_{i-n^{(p)}}^{(p)} & \frac{3}{2}n^{(p)} \leq i \leq 2n^{(p)} - 1 \end{cases} \tag{27}$$

It follows that column sequences of $\mathbf{C}^{(p+1)}$ are equal to, or are a negation of, column sequences of $\mathbf{C}^{(p)}$. Since column sequences of $\mathbf{C}^{(0)}$ are $\mathbf{c}_0$ and $\mathbf{c}_1$, all column sequences of $\mathbf{C}^{(p)}$, $p = 0, 1, 2, ...$, are in $\mathcal{R}^{(0)}_{\mathbf{c}_0,\mathbf{c}_1}$. In addition, for $\mathbf{C}^{(p+1)}$ constructed using the length-extension (16), the interleaving of the corresponding row sequences doesn't change the column sequences and, thus, its column sequences are also in $\mathcal{R}^{(0)}_{\mathbf{c}_0,\mathbf{c}_1}$. Any column sequence of $\mathbf{D}^{(p)}$ can be found in $\mathbf{C}^{(p+1)}$ and, consequently, it is in $\mathcal{R}^{(0)}_{\mathbf{c}_0,\mathbf{c}_1}$ as well. ■

Lemma 3.2: All column sequences of the constructed MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ are in $\mathcal{R}^{(t)}_{\mathbf{c}_0,\mathbf{c}_1}$, where $(\mathbf{c}_0, \mathbf{c}_1)$ is the companion pair and $t, p = 0, 1, 2, ...$

Proof: Let $\{\mathbf{u}^{(t)}_i, 0 \leq i < r = 2^{2t+1} n^{(p)}\}$ denote column sequences of $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$. The size-extension (17) implies,

$$\begin{cases} \mathbf{u}^{(t+1)}_i & = -\mathbf{u}^{(t+1)}_{i+2r} & = \mathbf{u}^{(t)}_i (-\mathbf{u}^{(t)}_i) \\ \mathbf{u}^{(t+1)}_{i+r} & = \mathbf{u}^{(t+1)}_{i+3r} & = \mathbf{u}^{(t)}_i \mathbf{u}^{(t)}_i \end{cases} \tag{28}$$

where $0 \leq i < r$. Based on *Lemma 3.1*, $\mathbf{u}^{(0)}_i$, i.e., column sequences of $\mathbf{M}^2_{m,n^{(p)}} = [\mathbf{C}^{(p)} \ \mathbf{D}^{(p)}]$, are in $\mathcal{R}^{(0)}_{\mathbf{c}_0,\mathbf{c}_1}$. From (12), (13), and (28), we have that

$$\mathbf{u}^{(t)}_i \in \mathcal{R}^{(t)}_{\mathbf{c}_0,\mathbf{c}_1}, \quad t = 0, 1, 2, ... \tag{29}$$

When $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ is constructed using size-extension (18), the proof is analogous. ■

IV. Properties of the constructed complementary set matrix

In this section, column correlation properties of the constructed MO complementary set matrix are related to ACFs of the companion pair. We illustrate how to satisfy a column correlation constraint by selecting an appropriate companion pair. We also construct the companion pair which leads to complementary set matrices with Golay column sequences. Since number of zeros in an energy-normalized sequence can affect its PAPR (see e.g. [6], [9]), we also discuss the number of zeros in column sequences at the end of this section.

A. Column correlation properties

Theorem 4.1: MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ satisfies a column correlation constraint, if and only if the companion pair $(\mathbf{c}_0, \mathbf{c}_1)$ is selected so that all sequences in $\mathcal{R}^{(t)}_{\mathbf{c}_0,\mathbf{c}_1}$ satisfy the constraint.

Proof: The proof is a direct consequence of *Lemma 3.2*. ■

Corollary 4.1: Complementary set matrices $\mathbf{C}^{(p)}$ and $\mathbf{D}^{(p)}$ constructed from a companion pair $(\mathbf{c}_0, \mathbf{c}_1)$, satisfy a column correlation constraint, if and only if $\mathbf{c}_0$ and $\mathbf{c}_1$ satisfy the constraint.

Proof: The proof follows by setting $t=0$ in *Theorem 4.1*. ■

The minimum achievable column correlation constraint for the constructed MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ is a function of its size and alphabet and can be expressed as follows,

$$\lambda_{min}(t,m) = \min\left\{ \max\left\{ \lambda_{\mathbf{c}} \ : \ \mathbf{c} \in \mathcal{R}^{(t)}_{\mathbf{c}_0,\mathbf{c}_1} \right\} : (\mathbf{c}_0,\mathbf{c}_1) \text{ is a companion pair and } \mathbf{c}_0,\mathbf{c}_1 \in \mathcal{X}(m) \right\}, \tag{30}$$

where λ is any autocorrelation merit. The following *Lemma 4.1* is a key in relating column correlation constraints to the correlation of the companion pair. In particular, it leads to the minimum achievable column correlation constraint $\lambda^A_{min}(t,m)$.

Lemma 4.1: The ACF of any column sequence $\mathbf{u}_i^{(t)}$ of $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ can be expressed in terms of the ACF of $\mathbf{c}_0$ or $\mathbf{c}_1$, recursively, as follows,

$$A_{\mathbf{u}_j^{(v+1)}}(l) = \begin{cases} 2A_{\mathbf{u}_i^{(v)}}(l) \pm A_{\mathbf{u}_i^{(v)}}(s-l), & 0 \le l < s; \\ \pm A_{\mathbf{u}_i^{(v)}}(l-s), & s \le l < 2s. \end{cases} \tag{31}$$

where '$+$' holds for $j=i+r$ or $j=i+3r$, '$-$' holds for $j=i$ or $j=i+2r$, when size-extension (17) is used; '$+$' holds for $j=2i+1$, '$-$' holds for $j=2i$, when size-extension (18) is employed; $\mathbf{u}_i^{(0)} \in \{\mathbf{c}_0,\mathbf{c}_1,-\mathbf{c}_0,-\mathbf{c}_1\}$; $0 \le i < r = 2^{2v+1}n^{(p)}$, $s=2^v m$, and $v=0,1,2,...,t-1$.

Proof: Eq. (31) can be derived based on (28). Note that, $\mathbf{u}_i^{(0)} \in \mathcal{R}^{(0)}_{\mathbf{c}_0,\mathbf{c}_1} = \{\mathbf{c}_0,\mathbf{c}_1,-\mathbf{c}_0,-\mathbf{c}_1\}$ and the negation of a sequence doesn't change its ACF. ■

Similar recursive equations can be found for periodic ACFs based on Eq. (3).

Proposition 4.1 (*A sufficient condition for* S^A): Let $S^A_{\mathbf{c}_0} \le S^A_0$, $S^A_{\mathbf{c}_1} \le S^A_0$, and $A_{\mathbf{c}_0}(0) = A_{\mathbf{c}_1}(0) = E$, where $(\mathbf{c}_0,\mathbf{c}_1)$ is a companion pair. Then, a sufficient condition for $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ to satisfy the column correlation constraint S^A_t is

$$S^A_t \ge 4^t S^A_0 + 2^{t-1}(2^t-1)E \ . \tag{32}$$

Proof: Let $\{\mathbf{u}_i^{(t)}, 0 \le i < 2^{2t+1}n^{(p)}\}$ be the column sequences of $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$. Clearly, $\mathbf{u}_i^{(0)} \in \{\mathbf{c}_0,\mathbf{c}_1,-\mathbf{c}_0,-\mathbf{c}_1\}$. Then, based on (31), we have that

$$\begin{aligned} S^A_{\mathbf{u}} &= \max_i \left\{ S^A_{\mathbf{u}_i^{(t)}}, 0 \le i < 2^{2t+1}n^{(p)} \right\} \\ &= \max_i \left\{ \sum_{l=1}^{2^t m-1} |A_{\mathbf{u}_i^{(t)}}(l)|, 0 \le i < 2^{2t+1}n^{(p)} \right\} \\ &\le 4^t S^A_0 + 2^{t-1}(2^t-1)E, \end{aligned} \tag{33}$$

for $t=0,1,2,....$ Hence, (32) is sufficient for $S^A_t \ge S^A_{\mathbf{u}}$ which proves the proposition. ■

Proposition 4.2 (*A sufficient condition for* λ^A): Let $\lambda^A_{\mathbf{c}_0} \le \lambda^A_0$, $\lambda^A_{\mathbf{c}_1} \le \lambda^A_0$, and $A_{\mathbf{c}_0}(0) = A_{\mathbf{c}_1}(0) = E$, where $(\mathbf{c}_0, \mathbf{c}_1)$ is a companion pair. Then, a sufficient condition for $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ to satisfy the column correlation constraint λ^A_t is

$$\lambda^A_t \ge \max\left\{(2^t - 1)E,\ (2^{t+1} - 1)\lambda^A_0\right\}. \tag{34}$$

Proof: Let $\{\mathbf{u}^{(t)}_i, 0 \le i < 2^{2t+1} n^{(p)}\}$ be the column sequences of $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$. (31) implies

$$\begin{aligned} \lambda^A_{\mathbf{u}} &= \max_i \left\{\lambda^A_{\mathbf{u}^{(t)}_i}, 0 \le i < 2^{2t+1} n^{(p)}\right\} \\ &= \max_{l,i} \left\{|A_{\mathbf{u}^{(t)}_i}(l)|, 1 \le l < 2^t m, 0 \le i < 2^{2t+1} n^{(p)}\right\} \\ &\le \max\left\{(2^t - 1)E, (2^{t+1} - 1)\lambda^A_0\right\} \end{aligned} \tag{35}$$

where $t = 0, 1, 2, ...$ Hence, if (34) holds, we have $\lambda^A_t \ge \lambda^A_{\mathbf{u}}$. ■

Proposition 4.3 (*A necessary condition for* λ^A): Let $A_{\mathbf{c}_0}(0) = A_{\mathbf{c}_1}(0) = E$, where $(\mathbf{c}_0, \mathbf{c}_1)$ is a companion pair. An achievable column correlation constraint λ^A_t of $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ must satisfy

$$\lambda^A_t \ge (2^t - 1)E. \tag{36}$$

Proof: (31) implies that $|A_{\mathbf{u}^{(t)}_k}(m)| = (2^t - 1)E$ for $k = 2^{2t+1} n^{(p)} - 1$. Hence, $\lambda^A_t \ge \max_{l,i}\{|A_{\mathbf{u}^{(t)}_i}(l)|, 1 \le l < 2^t m, 0 \le i < 2^{2t+1} n^{(p)}\} \ge (2^t - 1)E$. ■

Corollary 4.2: Let $A_{\mathbf{c}_0}(0) = A_{\mathbf{c}_1}(0) = E$ and $\lambda^A_0 = \max\{\lambda^A_{\mathbf{c}_0}, \lambda^A_{\mathbf{c}_1}\}$, where $(\mathbf{c}_0, \mathbf{c}_1)$ is a companion pair. For $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$, $t \ge 1$, when

$$\lambda^A_0 \le \frac{2^t - 1}{2^{t+1} - 1} E, \tag{37}$$

the minimum column correlation constraint

$$\lambda^A_{min}(t, m) = (2^t - 1)E \tag{38}$$

is achievable.

Proof: If (37) holds, based on *Proposition 4.2*, $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ satisfies the column correlation constraint $\lambda^A_t = (2^t - 1)E$. On the other hand, *Proposition 4.3* states that $\lambda^A_t \ge (2^t - 1)E$ must hold. ■

Example 4.1: To construct the complex-valued complementary set matrices $\mathbf{C}^{(p)}$ and $\mathbf{D}^{(p)}$ with a column correlation constraint $\lambda^A = 1$, we can choose a companion pair $\mathbf{c}_0 = (+, j, -, j)$ and $\mathbf{c}_1 = f^*_i(\mathbf{c}_0) = (\bar{j}, -, \bar{j}, +)$, where $+$ denotes 1, $-$ denotes -1, $j = \sqrt{-1}$ denotes the imaginary unit and $\bar{j}$ denotes $-j$, which satisfy $\lambda^A_{\mathbf{c}_i} \le 1$,

$i = 0, 1$. Then, the companion matrix is

$$\mathbf{C}^{(0)} = \begin{bmatrix} + & j & - & j \\ \bar{j} & - & \bar{j} & + \end{bmatrix}^T. \tag{40}$$

By employing length-extension (15), the complementary set matrix $\mathbf{C}^{(1)}$ and its mate $\mathbf{D}^{(1)}$ can be obtained as

$$\mathbf{C}^{(1)} = \begin{bmatrix} + & \bar{j} & - & \bar{j} \\ j & - & \bar{j} & - \\ - & \bar{j} & + & \bar{j} \\ j & + & \bar{j} & + \end{bmatrix}_{4\times 4}, \ \mathbf{D}^{(1)} = \begin{bmatrix} - & j & - & \bar{j} \\ \bar{j} & + & \bar{j} & - \\ + & j & + & \bar{j} \\ \bar{j} & - & \bar{j} & + \end{bmatrix}_{4\times 4}. \tag{41}$$

Based on *Corollary 4.1*, $\mathbf{C}^{(1)}$ and $\mathbf{D}^{(1)}$ are complementary set matrices with a column correlation constraint $\lambda^A = 1$.

Example 4.2: Let again $\mathbf{c}_0 = (+, j, -, j)$ and $\mathbf{c}_1 = f_i^*(\mathbf{c}_0) = (\bar{j}, -, \bar{j}, +)$, then, any sequence $\mathbf{c}$ in $\mathcal{R}^{(1)}_{\mathbf{c}_0,\mathbf{c}_1}$ satisfies $\lambda^A_{\mathbf{c}} \le \lambda^A = 4$. Starting with $\mathbf{M}^2_{4,4} = [\mathbf{C}^{(1)}\ \mathbf{D}^{(1)}]$ constructed in *Example 4.1* and applying the size-extension (17), we obtain $\mathbf{M}^4_{8,8}$ as shown in (39). Based on *Theorem 4.1*, $\mathbf{M}^4_{8,8}$ satisfies a column correlation constraint $\lambda^A = 4$. On the other hand, *Corollary 4.2* implies that $\mathbf{M}^4_{8,8}$ achieves its lower bound $\lambda^A_1 = E = 4$, since $\lambda^A_0 = 1 \le \frac{E}{3}$.

Example 4.3: Let us consider how to construct $\mathbf{M}^4_{8,8}$ satisfying a column correlation constraint $S^A = 12$. Since $t = 1$ and $m = 4$, based on *Proposition 4.1*, a sufficient condition for $S^A_1 = 12$ is $S^A_0 \le 2$, for $E = 4$. The companion pair $(\mathbf{c}_0, \mathbf{c}_1)$ in *Examples 4.1-4.2* satisfies $S^A_{\mathbf{c}_i} \le S^A_0 = 2$, for $i = 0, 1$. Thus, $\mathbf{M}^4_{8,8}$ in (39) must also satisfy a column correlation constraint $S^A = 12$. Let $\{\mathbf{u}^{(1)}_i, 0 \le i \le 31\}$ denote column sequences of $\mathbf{M}^4_{8,8}$ in (39), we can verify that $\max\left\{S^A_{\mathbf{u}^{(1)}_i}, 0 \le i \le 31\right\} = 12$.

$$\mathbf{M}^4_{8,8} = \left[\begin{array}{c|c|c|c}
+\bar{j}-\bar{j}-j-\bar{j} & +\bar{j}-\bar{j}-j-\bar{j} & -j+j+\bar{j}+j & +\bar{j}-\bar{j}-j-\bar{j} \\
j-\bar{j}-\bar{j}+\bar{j}- & j-\bar{j}-\bar{j}+\bar{j}- & \bar{j}+j+j-j+ & j-\bar{j}-\bar{j}+\bar{j}- \\
-\bar{j}+\bar{j}+j+\bar{j} & -\bar{j}+\bar{j}+j+\bar{j} & +j-j-\bar{j}-j & -\bar{j}+\bar{j}+j+\bar{j} \\
j+\bar{j}+\bar{j}-\bar{j}+ & j+\bar{j}+\bar{j}-\bar{j}+ & \bar{j}-j-j+j- & j+\bar{j}+\bar{j}-\bar{j}+ \\
-j+j+\bar{j}+j & +\bar{j}-\bar{j}-j-\bar{j} & +\bar{j}-\bar{j}-j-\bar{j} & +\bar{j}-\bar{j}-j-\bar{j} \\
\bar{j}+j+j-j+ & j-\bar{j}-\bar{j}+\bar{j}- & j-\bar{j}-\bar{j}+\bar{j}- & j-\bar{j}-\bar{j}+\bar{j}- \\
+j-j-\bar{j}-j & -\bar{j}+\bar{j}+j+\bar{j} & -\bar{j}+\bar{j}+j+\bar{j} & -\bar{j}+\bar{j}+j+\bar{j} \\
\bar{j}-j-j+j- & j+\bar{j}+\bar{j}-\bar{j}+ & j+\bar{j}+\bar{j}-\bar{j}+ & j+\bar{j}+\bar{j}-\bar{j}+
\end{array}\right]_{8\times 32} \tag{39}$$

Remark: For the case $t = 0$, *Corollary 4.1* implies that the correlation constraint for the companion pair is also the column correlation constraint of $\mathbf{M}^2_{m,n^{(p)}} = [\mathbf{C}^{(p)}\ \mathbf{D}^{(p)}]$. For $t \geq 1$, based on *Proposition 4.1*, small S_0^A may also help in reducing the column correlation constraint S_t^A. However, *Corollary 4.2* implies that it is not necessary to search for the companion pair with smaller λ_0^A, once the lower bound $\lambda_{min}^A(t, m) = (2^t - 1)E$ has been achieved.

B. Golay column sequences

Theorem 4.2: Column sequences of complementary set matrices $\mathbf{C}^{(p)}$ and $\mathbf{D}^{(p)}$ are Golay sequences, if and only if the companion sequences $\mathbf{c}_0$ and $\mathbf{c}_1$ are both Golay.

Proof: Lemma 3.1 states that column sequences of $\mathbf{C}^{(p)}$ and $\mathbf{D}^{(p)}$ are either $\pm\mathbf{c}_0$ or $\pm\mathbf{c}_1$. Note that, a negation of a Golay sequence is also a Golay sequence. ■

We present a constructive method to obtain the companion pair from which an m by n complementary set matrix with Golay column sequences can be constructed, where $m = 2^{q+1}$, $n = 2^{p+1}$, $p, q = 0, 1, 2....$

Theorem 4.3: Let

$$\mathbf{H}_i^{(q)} = \begin{bmatrix} \mathbf{h}_{i,0}^{(q)} \\ \mathbf{h}_{i,1}^{(q)} \end{bmatrix}_{2\times 2^q} = \begin{bmatrix} \mathbf{h}_{i,0}^{(q-1)} \overleftarrow{\mathbf{h}_{i,1}^{(q-1)}} \\ \mathbf{h}_{i,1}^{(q-1)} (-\overleftarrow{\mathbf{h}_{i,0}^{(q-1)}}) \end{bmatrix}, \tag{42}$$

where $\mathbf{h}_{i,0}^{(q)}$ and $\mathbf{h}_{i,1}^{(q)}$ are two row sequences of $\mathbf{H}_i^{(q)}$, $i = 0, 1$. The initial matrices are

$$\mathbf{H}_0^{(0)} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}_{2\times 2}, \quad \mathbf{H}_1^{(0)} = \begin{bmatrix} + & - \\ + & + \end{bmatrix}_{2\times 2}. \tag{43}$$

Then, $\{\mathbf{h}_{0,0}^{(q)}, \mathbf{h}_{0,1}^{(q)}\}$ and $\{\mathbf{h}_{1,0}^{(q)}, \mathbf{h}_{1,1}^{(q)}\}$ are respectively Golay complementary pairs and, furthermore, $f_i(\mathbf{h}_{0,0}^{(q)}) = \mathbf{h}_{1,0}^{(q)}$ and $f_i(\mathbf{h}_{0,1}^{(q)}) = -\mathbf{h}_{1,1}^{(q)}$, for $q = 0, 1, 2....$

Proof: See Appendix C. ■

Example 4.4: Let $q = 2$, then $\mathbf{c}_0 = \mathbf{h}_{0,0}^{(2)} = (++-+---+)$ and $\mathbf{c}_1 = \mathbf{h}_{1,0}^{(2)} = (+--++-++)$. Based on *Theorem 4.3*, $\{\mathbf{c}_0, \mathbf{h}_{0,1}^{(2)}\}$ and $\{\mathbf{c}_1, \mathbf{h}_{1,1}^{(2)}\}$ are, respectively, Golay complementary pairs, where $\mathbf{h}_{0,1}^{(2)} = (+----+--)$ and $\mathbf{h}_{1,1}^{(2)} = (+++---+-)$. Thus, the companion sequences $\mathbf{c}_0$ and $\mathbf{c}_1$ are Golay sequences. The length-extension (16) for $p=2$ allows for constructing the following complementary set matrix

$$\mathbf{C}^{(2)} = \begin{bmatrix} + & - & - & - & + & - & + & + \\ + & - & - & - & - & + & - & - \\ - & + & + & + & + & - & + & + \\ + & - & - & - & + & - & + & + \\ - & + & + & + & - & + & - & - \\ - & + & + & + & + & - & + & + \\ - & + & + & + & + & - & + & + \\ + & - & - & - & + & - & + & + \end{bmatrix}_{8\times 8} \tag{44}$$

whose column sequences are Golay. Hence, the PAPR of all column sequences of $\mathbf{C}^{(2)}$ is at most two [17].

C. Number of zeros

Proposition 4.4: Let the companion sequence $\mathbf{c}_0$ be a length m sequence with z zeros, then, any column sequence of $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ contains $2^t z$ zeros.

Proof: Based on Eqs. (21) and (22), $\mathbf{c}_0$ and its companion $\mathbf{c}_1$ have the same number of zeros. The number of zeros in each column sequence does not change after each length-extension operation. Each size-extension operation doubles the length of column sequences, as well as the number of zeros. ■

Example 4.5: In this example, we consider a ternary complementary set matrix and its mate with a column correlation constraint $S^A = 5$. Let us set $m = 8$ and $z = 1$. We can find the companion pair $\mathbf{c}_0 = (+--+++0+)$ and $\mathbf{c}_1 = f_i(\mathbf{c}_0) = (--+++-+0)$ which satisfy $S^A_{\mathbf{c}_i} \leq 5$, $i = 0, 1$. The companion matrix is

$$\mathbf{C}^{(0)} = \begin{bmatrix} +--+++\ 0+ \\ --+++-+0 \end{bmatrix}^T . \tag{45}$$

Using length-extension (15), we extend $\mathbf{C}^{(0)}$ as

$$\mathbf{C}^{(2)} = \begin{bmatrix} + & - & - & - & - & + & - & - \\ - & - & + & - & + & + & + & - \\ - & + & + & + & + & - & + & + \\ + & + & - & + & - & - & - & + \\ + & + & - & + & - & - & - & + \\ + & - & - & - & - & + & - & - \\ 0 & + & 0 & + & 0 & - & 0 & + \\ + & 0 & - & 0 & - & 0 & - & 0 \end{bmatrix}_{8\times 8} \quad \text{and} \quad \mathbf{D}^{(2)} = \begin{bmatrix} - & + & + & + & - & + & - & - \\ + & + & - & + & + & + & + & - \\ + & - & - & - & + & - & + & + \\ - & - & + & - & - & - & - & + \\ - & - & + & - & - & - & - & + \\ - & + & + & + & - & + & - & - \\ 0 & - & 0 & - & 0 & - & 0 & + \\ - & 0 & + & 0 & - & 0 & - & 0 \end{bmatrix}_{8\times 8}. \tag{46}$$

Based on *Theorem 4.1*, complementary set matrices $\mathbf{C}^{(2)}$ and $\mathbf{D}^{(2)}$ satisfy the column correlation constraint $S^A = 5$. Furthermore, by setting $t = 0$ in *Proposition 4.4*, we have that any column sequence of $\mathbf{C}^{(p)}$ and $\mathbf{D}^{(p)}$ has only one zero.

V. Search for companion pairs

A. Exhaustive search algorithm

Let $\mathcal{X}(n, \lambda)$ denote a subset of all sequences of $\mathcal{X}(n)$ which satisfy the correlation constraint λ. For example, let $\mathcal{B}(2^t m) = \{\mathbf{c} \,|\, c_i \in \mathcal{B}, 1 \le i \le 2^t m\}$, then $\mathcal{B}(2^t m, S^A) = \{\mathbf{c} \,|\, \mathbf{c} \in \mathcal{B}(2^t m), S^A_{\mathbf{c}} \le S^A\}$, where $\mathcal{B} = \{+1, -1\}$. Clearly, all column sequences of binary MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ with a column correlation constraint S^A can be found in $\mathcal{B}(2^t m, S^A)$. Hence, to construct a MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ whose column sequences are in $\mathcal{X}(n, \lambda)$, we need to find a companion pair $(\mathbf{c}_0, \mathbf{c}_1)$ such that $\mathcal{R}^{(t)}_{\mathbf{c}_0, \mathbf{c}_1} \subseteq \mathcal{X}(n, \lambda)$ (see *Lemma 3.2*).

Let us index all $K = |\mathcal{X}(m)|$ sequences as $\mathbf{x}_i$, for $1 \le i \le K$. When a column correlation constraint λ is given, desired companion pairs can be obtained by exhaustive computer search over $\mathcal{X}(m)$, as described in Table III.

Note that the ACFs of sequences in $\mathcal{R}^{(t)}_{\mathbf{x}_i}$ can be computed recursively using (31). For binary sequences, we can simply check if $\mathbf{x} \cdot \mathbf{y} = 0$ to determine the companion pair.

B. Minimum achievable column correlation constraint

The exhaustive search algorithm in Table III can be easily modified to search for the companion pair with a minimum achievable column correlation constraint. However, the computing load is heavy, especially for large m

TABLE III

EXHAUSTIVE SEARCH ALGORITHM

```
j=0;
for   i = 1, 2, 3, ..., K   loop
    if   R_{x_i}^{(t)} ⊆ X(2^t m, λ),
        j = j + 1; y_j = x_i;
        for   l = j − 1, j − 2, ...1,
            check if (y_j, y_l) is a companion pair;
        end
    end
end   loops
```

and t. Let $t = 0$ in (30), the companion pair for the construction of $\mathbf{M}^2_{m,n^{(p)}} = [\mathbf{C}^{(p)}\ \mathbf{D}^{(p)}]$ with a minimum achievable column correlation constraint is

$$(\mathbf{c}_0, \mathbf{c}_1) = \arg\min_{(\mathbf{x},\mathbf{y})} \{\max\{\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}\} : (\mathbf{x}, \mathbf{y}) \text{ is a companion pair and } \mathbf{x}, \mathbf{y} \in \mathcal{X}(m)\}. \tag{47}$$

Based on *Propositions 4.1-4.3*, the above companion pair may also lead to the MO complementary set matrix $\mathbf{M}^{2^{t+1}}_{2^t m, 2^t n^{(p)}}$ with a reduced column correlation constraint for $t \geq 1$. Hence, in the following, we consider companion pairs leading to an achievable or a minimum achievable column correlation constraint $\lambda_{min}(m) = \lambda_{min}(t = 0, m)$ for the case $t = 0$ only. Table IV lists binary companion pairs with minimum achievable column correlation constraints $\lambda^A_{min}(m)$ and $S^A_{min}(m)$. It can be observed that most of these companion pairs can achieve $\lambda^A_{min}(m)$ and $S^A_{min}(m)$ simultaneously.

When m is large, the exhaustive computer search is infeasible. Hence, the existence of a companion pair satisfying a correlation constraint is an important problem considered in the next section.

VI. THE EXISTENCE OF COMPANION PAIRS

The existence of a sequence of a desired correlation constraint has been studied in literature. For example, binary sequences with $\lambda^A = 1$ exist only for lengths $2, 3, 4, 5, 7, 11$ and 13, and are called binary Barker sequences; binary m-sequences [23] with $\lambda^P = 1$ exist for length $m = 2^l - 1$, $l = 2, 3, 4....$ In this section, we exploit the correlation properties of companion pairs and analyze their existence for correlation constraints λ^A and λ^P.

TABLE IV

BINARY COMPANION PAIRS

m	λ^A_{min}	S^A_{min}	companion pair	m	λ^A_{min}	S^A_{min}	companion pair
2	1	1	$-+$ $++$	12	2	8	$++++-+-++--+$ $--++-----+-+$
4	1	2	$---+$ $-+++$	14	2	13	$+-+------++--+$ $----++--++-+-+$
6	2	5	$-+---+$ $--+-++$	16	2	/	$-+++------+--+-+$ $+--++-+-+------+$
8	2	6	$-+----++$ $+++--+-+$	16	/	12	$++--+++++-+-+--+$ $-+++-+++-+--+-++$
10	2	9	$+-+++++--+$ $---++--+-+$	18	2	17	$++++----+-+--++--+$ $-+-++-+-----++--++$

A. *Correlation properties of the companion pair constructed using two arbitrary sequences*

In the following Cases 1 and 2 we illustrate how a companion pair of length m can be formed using two arbitrary sequences $\mathbf{s}_0$ and $\mathbf{s}_1$ of length $m/2$. We study the correlation properties of the companion pair $(\mathbf{c}_0, \mathbf{c}_1)$ constructed from $\mathbf{s}_0$ and $\mathbf{s}_1$.

Case 1: Let $\mathbf{c}_0 = \mathbf{s}_0 \otimes \mathbf{s}_1$ and $\mathbf{c}_1 = \mathbf{s}_1^* \otimes (-\mathbf{s}_0^*)$. Then $\mathbf{c}_1$ is a companion of $\mathbf{c}_0$ since $\mathbf{c}_1 = f_i^*(\mathbf{c}_0)$.

In this case, ACFs of the companion pair can be expressed in terms of the ACFs of $\mathbf{s}_0$ and $\mathbf{s}_1$ and their crosscorrelation functions,

$$A_{\mathbf{c}_0}(l) = \begin{cases} A_{\mathbf{s}_0,\mathbf{s}_1}(\frac{l-1}{2}) + A_{\mathbf{s}_0,\mathbf{s}_1}(\frac{-l-1}{2}), & l \in odd \\ A_{\mathbf{s}_0}(\frac{l}{2}) + A_{\mathbf{s}_1}(\frac{l}{2}), & l \in even \end{cases} \tag{48}$$

$$A_{\mathbf{c}_1^*}(l) = \begin{cases} -A_{\mathbf{s}_0,\mathbf{s}_1}(\frac{l+1}{2}) - A_{\mathbf{s}_0,\mathbf{s}_1}(\frac{-l+1}{2}), & l \in odd \\ A_{\mathbf{s}_0}(\frac{l}{2}) + A_{\mathbf{s}_1}(\frac{l}{2}), & l \in even \end{cases} \tag{49}$$

$$P_{\mathbf{c}_0}(l) = \begin{cases} P_{\mathbf{s}_0,\mathbf{s}_1}(\frac{l-1}{2}) + P_{\mathbf{s}_0,\mathbf{s}_1}(\frac{m-l-1}{2}), & l \in odd \\ P_{\mathbf{s}_0}(\frac{l}{2}) + P_{\mathbf{s}_1}(\frac{l}{2}), & l \in even \end{cases} \tag{50}$$

$$P_{\mathbf{c}_1^*}(l) = \begin{cases} -P_{\mathbf{s}_0,\mathbf{s}_1}(\frac{l+1}{2}) - P_{\mathbf{s}_0,\mathbf{s}_1}(\frac{m-l+1}{2}), & l \in odd \\ P_{\mathbf{s}_0}(\frac{l}{2}) + P_{\mathbf{s}_1}(\frac{l}{2}), & l \in even \end{cases} \tag{51}$$

where $0 \leq l \leq m-1$.

Lemma 6.1: Let $\mathbf{c}_0 = \mathbf{s}_0 \otimes \mathbf{s}_1$ and $\mathbf{c}_1^* = \mathbf{s}_1 \otimes (-\mathbf{s}_0)$, then

$$\begin{cases} \lambda_{\mathbf{c}_i}^A \leq \max\{\lambda_{\mathbf{s}_0}^A + \lambda_{\mathbf{s}_1}^A, 2\lambda_{\mathbf{s}_0,\mathbf{s}_1}^A\} \\ \lambda_{\mathbf{c}_i}^P \leq \max\{\lambda_{\mathbf{s}_0}^P + \lambda_{\mathbf{s}_1}^P, 2\lambda_{\mathbf{s}_0,\mathbf{s}_1}^P\} \end{cases} \tag{52}$$

and

$$\begin{cases} S_{\mathbf{c}_i}^A \leq S_{\mathbf{s}_0}^A + S_{\mathbf{s}_1}^A + S_{\mathbf{s}_0,\mathbf{s}_1}^A \\ S_{\mathbf{c}_i}^P \leq S_{\mathbf{s}_0}^P + S_{\mathbf{s}_1}^P + 2S_{\mathbf{s}_0,\mathbf{s}_1}^P \end{cases} \tag{53}$$

where $i = 0, 1$.

Proof: See Appendix C. ■

Case 2: Let $\mathbf{c}_0 = \mathbf{s}_0\mathbf{s}_1$ and $\mathbf{c}_1 = \mathbf{s}_1^*(-\mathbf{s}_0^*)$. Then $\mathbf{c}_1$ is a companion of $\mathbf{c}_0$ since $\mathbf{c}_1 = f_c^*(\mathbf{c}_0)$.

The aperiodic ACFs of the companion pair can be expressed as

$$A_{\mathbf{c}_0}(l) = \begin{cases} A_{\mathbf{s}_0}(l) + A_{\mathbf{s}_1}(l) + A_{\mathbf{s}_0,\mathbf{s}_1}(l - \frac{m}{2}), & 0 \leq l < \frac{m}{2} \\ A_{\mathbf{s}_0,\mathbf{s}_1}(l - \frac{m}{2}) & \frac{m}{2} \leq l < m \end{cases} \tag{54}$$

$$A_{\mathbf{c}_1}(l) = \begin{cases} A_{\mathbf{s}_0}(l) + A_{\mathbf{s}_1}(l) - A_{\mathbf{s}_0,\mathbf{s}_1}(\frac{m}{2} - l), & 0 \leq l < \frac{m}{2} \\ -A_{\mathbf{s}_0,\mathbf{s}_1}(\frac{m}{2} - l) & \frac{m}{2} \leq l < m \end{cases} \tag{55}$$

Lemma 6.2: Let $\mathbf{c}_0 = \mathbf{s}_0\mathbf{s}_1$ and $\mathbf{c}_1^* = \mathbf{s}_1(-\mathbf{s}_0)$, then

$$\lambda_{\mathbf{c}_i}^A \leq \lambda_{\mathbf{s}_0}^A + \lambda_{\mathbf{s}_1}^A + \lambda_{\mathbf{s}_0,\mathbf{s}_1}^A, \quad i = 0, 1. \tag{56}$$

Proof: The proof is along the lines of the proof of *Lemma 6.1*. ■

B. Existence

Without loss of generality, we assume that $\mathbf{s}_i$ are complex-valued sequences of length $m/2$ and $A_{\mathbf{s}_i}(0) = P_{\mathbf{s}_i}(0) = m/2$, $i = 0, 1$.

Lemma 6.3 (Welch bound [25]): Let $\{\mathbf{s}_i, i = 0, 1, ..., K-1\}$, denote a set of K complex-valued sequences of length N. If $A_{\mathbf{s}_i}(0) = P_{\mathbf{s}_i}(0) = N$ for all i, then,

$$P_{max} \geq N\sqrt{\frac{K-1}{NK-1}} \tag{57}$$

$$A_{max} \geq N\sqrt{\frac{K-1}{2NK-K-1}} \tag{58}$$

where

$$P_{max} = \max_{0 \leq i,j < K, i \neq j}\{\lambda_{\mathbf{s}_i}^P, \lambda_{\mathbf{s}_i,\mathbf{s}_j}^P\} \tag{59}$$

$$A_{max} = \max_{0 \leq i,j < K, i \neq j}\{\lambda_{\mathbf{s}_i}^A, \lambda_{\mathbf{s}_i,\mathbf{s}_j}^A\}. \tag{60}$$

Proof: The proof can be found in [25]. ■

1) Column correlation constraint λ^A: The following *Theorems 6.1-6.2* restate the companion pair existence conditions from *Theorems 4.1-4.2* in terms of the $\{\mathbf{s}_0, \mathbf{s}_1\}$ pair existence conditions from *Lemmas 6.1-6.2*.

Theorem 6.1: MO complementary set matrix $\mathbf{M}^2_{m,n^{(p)}}$ with a column correlation constraint λ^A exists if there exists a sequence pair $\{\mathbf{s}_0, \mathbf{s}_1\}$ with $A_{max} = \frac{1}{2}\lambda^A$.

Proof: Theorem 4.1 states that MO complementary set matrix $\mathbf{M}^2_{m,n^{(p)}}$ satisfying the column correlation constraint λ^A exists, if and only if we can find a companion pair $(\mathbf{c}_0, \mathbf{c}_1)$, such that,

$$\lambda^A_{\mathbf{c}_i} \le \lambda^A, \quad i = 0, 1 \tag{61}$$

Based on (52), a sufficient condition for (61) is,

$$\lambda^A_{\mathbf{s}_i} \le \frac{\lambda^A}{2}, \ i = 0, 1 \ \text{ and } \ \lambda^A_{\mathbf{s}_0,\mathbf{s}_1} \le \frac{\lambda^A}{2} \tag{62}$$

Based on (56),

$$\lambda^A_{\mathbf{s}_i} \le \frac{\lambda^A}{3}, \ i = 0, 1 \ \text{ and } \ \lambda^A_{\mathbf{s}_0,\mathbf{s}_1} \le \frac{\lambda^A}{3} \tag{63}$$

Hence, by comparing (62) and (63), we can set $A_{max} = \frac{1}{2}\lambda^A$. ■

Proposition 6.1: Sequence pair $\{\mathbf{s}_0, \mathbf{s}_1\}$ of length $\frac{m}{2}$ with $A_{max} = \frac{1}{2}\lambda^A$ exists only if

$$\lambda^A \ge \frac{m}{\sqrt{2m-3}} \tag{64}$$

Proof: Let $A_{max} = \frac{1}{2}\lambda^A$, $K = 2$, and $N = \frac{m}{2}$ in (58) of *Lemma 6.3*, then (64) follows. ■

Corollary 6.1: Let $\mathbf{c}_0 = \mathbf{s}_0 \otimes \mathbf{s}_1$ and $\mathbf{c}_1 = \mathbf{s}_1 \otimes (-\mathbf{s}_0)$ be a binary companion pair of length m, and $\mathbf{u}_i$ denote column sequences of the constructed $\mathbf{M}^2_{m,n^{(p)}}$, $0 \le i < 2n^{(p)}$. Then, $\lambda^A_W \le \lambda^A_{\mathbf{u}} \le \lambda^A_B$, where

$$\lambda^A_{\mathbf{u}} = \max_i \left\{ \lambda^A_{\mathbf{u}_i^{(t)}}, 0 \le i < 2n^{(p)} \right\} \tag{65}$$

$$\lambda^A_B = \max\left\{ \lambda^A_{\mathbf{s}_0} + \lambda^A_{\mathbf{s}_1}, 2\lambda^A_{\mathbf{s}_0,\mathbf{s}_1} \right\}, \tag{66}$$

$$\lambda^A_W = \lceil \frac{m}{\sqrt{2m-3}} \rceil. \tag{67}$$

Proof: λ^A_W is derived from *Theorem 6.1* and (64) by noting that $\lambda^A_{\mathbf{u}}$ is an integer for binary sequences. λ^A_B follows from *Lemma 6.1*. ■

2) Column correlation constraint: λ^P

Theorem 6.2: MO complementary set matrix $\mathbf{M}^2_{m,n^{(p)}}$ with a column correlation constraint λ^P exists, if there exists $\{\mathbf{s}_0, \mathbf{s}_1\}$ of length $m/2$ with $P_{max} = \frac{1}{2}\lambda^P$.

Proof: The proof follows along the lines of the proof of *Theorem 6.1* and is omitted. ■

Proposition 6.2: A length $m/2$ sequence pair $\{\mathbf{s}_0, \mathbf{s}_1\}$ with $P_{max} = \frac{1}{2}\lambda^P$ exists only if,

$$\lambda^P \geq \frac{m}{\sqrt{m-1}} \tag{68}$$

Proof: Setting $P_{max} = \frac{1}{2}\lambda^P$, $K = 2$ and $N = \frac{m}{2}$ in (57) leads to (68). ■

C. Achievable column correlation constraints

Theorems 6.1-6.2 suggest searching for sequences $\mathbf{s}_0$ and $\mathbf{s}_1$ of length $m/2$ with good autocorrelation and crosscorrelation merits to form a companion pair with a small achievable column correlation constraint. Former is a long standing problem (e.g. see [18]–[21]). In [20], good binary sequence pairs with small $\lambda^A_{\mathbf{s}_0}$, $\lambda^A_{\mathbf{s}_1}$ and $\lambda^A_{\mathbf{s}_0,\mathbf{s}_1}$ were found by using simulated annealing search algorithm, and were listed in Tables I and II. Based on *Corollary 6.1*, we present λ^A_W and λ^A_B of their corresponding binary companion pairs in Table V, where the reference [20] indicates that data is obtained by using sequences from this reference. However, the cost function for the simulated annealing in [20] is not optimal in our case. We instead minimize the cost function

$$f(\mathbf{s}_0, \mathbf{s}_1) = \max\left\{\lambda^A_{\mathbf{s}_0} + \lambda^A_{\mathbf{s}_1}, 2\lambda^A_{\mathbf{s}_0,\mathbf{s}_1}\right\} \tag{69}$$

to obtain an improved λ^A_B.

In Table VI, sequence pairs $\{\mathbf{s}_0, \mathbf{s}_1\}$ of length $m/2 = 63, 84$ and 100 obtained using simulated annealing based on (69) are presented. The corresponding ACF merit $\lambda^A_{\mathbf{u}}$ is calculated and compared to that of the sequence pairs from [20]. The proposed sequence pairs lead to companion pairs with an improved autocorrelation correlation merit.

TABLE V

λ^A_W AND λ^A_B FOR LONG BINARY SEED SEQUENCES

m	62	74	82	106	118	122	126	134	146	158	168	182	186	200	218	240
λ^A_W	6	7	7	8	8	8	8	9	9	9	10	10	10	11	11	11
λ^A_B [20]	16	18	16	18	20	22	22	22	24	24	28	24	24	28	30	28
λ^A_B	13	15	15	18	18	18	19	20	22	22	24	24	24	27	28	28

VII. CONCLUSION

We have considered a construction algorithm for MO complementary set matrices satisfying a column correlation constraint. The algorithm recursively constructs the MO complementary set matrix, starting from a companion pair.

We relate correlation properties of column sequences to that of the companion pair and illustrate how to select an appropriate companion pair to satisfy a given column correlation constraint. We also reveal a method to construct the Golay companion pair which leads to the complementary set matrix with Golay column sequences. An exhaustive computer search algorithm is described which helps in searching for companion pairs with a minimum achievable column correlation constraint. Exhaustive search is infeasible for relatively long sequences. Hence, we instead suggest a strategy for finding companion pairs with a small, if not minimum, column correlation constraint. Based on properties of the companion pair, the strategy suggests a search for any two shorter sequences by minimizing a cost function in terms of their autocorrelation and crosscorrelation merits, from which the desired companion pair can be formed. An improved cost function is derived to further reduce the achievable column correlation constraint λ^A. By exploiting the well-known Welch bound, sufficient conditions for the existence of companion pairs are also derived for column correlation constraints λ^A and λ^P.

We have left the general problem of finding MO complementary set matrices with a minimum column correlation constraint as an open question. An important step towards solving the general problem is to find new construction approaches for MO complementary set matrices. A design algorithm based on N-shift cross-orthogonal sequences can be found in [24]. However, their column correlation properties are intractable.

TABLE VI

ACHIEVABLE λ^A FOR LONG BINARY SEED SEQUENCES

m	merits	$\mathbf{s}_0$ (or $\mathbf{s}_1$)	$\mathbf{s}_1$ (or $\mathbf{s}_0$)
126	$\lambda_B^A = 19$ $\lambda_{\mathbf{u}}^A = 17$ $\lambda_B^A = 22$ [20] $\lambda_{\mathbf{u}}^A = 17$ [20]	+ − + − + − − + − + − + + + + + + + − − + − − + − + + + − − + − − − + + + − − − + + − − + + − + + − − − − + + − + − − − + + +	+ + − + + + − + + − − + + − − − − + − + + − + + + − − + + + + − − + + + + + − + − + + + + − + − − − − + + − + − + − + − − − +
168	$\lambda_B^A = 24$ $\lambda_{\mathbf{u}}^A = 20$ $\lambda_B^A = 28$ [20] $\lambda_{\mathbf{u}}^A = 21$ [20]	+ − − + − − + + + + + − − − − + − + + − − + − + + − − + + − − + + − + + − − − − + + − + + + − + − + − − − + + + − + + + + − − + − + + − + − + + − − − − − − + − − − − −	+ − − − + − − − − − − + + − − − + + + + − − − + − − − + + − + + + − − − + − + − + + − + − + − − + + + + + − − − + + + + + + − + + − + + − + + + − + − + + − + − − + − −
200	$\lambda_B^A = 27$ $\lambda_{\mathbf{u}}^A = 23$ $\lambda_B^A = 28$ [20] $\lambda_{\mathbf{u}}^A = 25$ [20]	− + + + + + − + + + + + − − − − + − + + + + − − − + + − + − + − + + − − + − + + + + + + − − − + + + + − + − + − − + − + − + + + + + − + + − − − + − + + − − + − − + − − + + + + − − − + − + − + − + + +	− − + + + − + + − + − − − + + + − − + + + − + + − − + − + − − + − − + − + − − − − − − − − + + − − + − − − − + + + − − + − + + + + + − − − − + + − − + − + + + − − + + − + − + + − + + − + − + + + − − +

Appendix

A. Proof of Lemma 2.1

Let us first prove that $\{\overleftarrow{\mathbf{b}^*}, -\overleftarrow{\mathbf{a}^*}\}$ is a mate of $\{\mathbf{a}, \mathbf{b}\}$. A proof for binary sequences can be found in *Theorem 11* of [3]. For complex-valued sequences, the complementarity of $\{\overleftarrow{\mathbf{b}^*}, -\overleftarrow{\mathbf{a}^*}\}$ follows from

$$\begin{aligned} A_{\overleftarrow{\mathbf{b}^*}}(l) + A_{-\overleftarrow{\mathbf{a}^*}}(l) &= (A_{\overleftarrow{\mathbf{b}}}(l))^* + (A_{-\overleftarrow{\mathbf{a}}}(l))^* \\ &= A_{\mathbf{b}}(l) + A_{-\mathbf{a}}(l) \\ &= A_{\mathbf{b}}(l) + A_{\mathbf{a}}(l) \\ &= 0 \end{aligned}$$

for $1 \leq l \leq n-1$, where n denotes the sequence length. We further show that the pair $\{\overleftarrow{\mathbf{b}^*}, -\overleftarrow{\mathbf{a}^*}\}$ is orthogonal to $\{\mathbf{a}, \mathbf{b}\}$ in the complementary sense, as follows

$$\begin{aligned} A_{\mathbf{a},\overleftarrow{\mathbf{b}^*}}(l) + A_{\mathbf{b},-\overleftarrow{\mathbf{a}^*}}(l) &= A_{\mathbf{a},\overleftarrow{\mathbf{b}^*}}(l) - A_{\mathbf{b},\overleftarrow{\mathbf{a}^*}}(l) \\ &= (A_{\mathbf{a}^*,\overleftarrow{\mathbf{b}}}(l))^* - A_{\mathbf{b},\overleftarrow{\mathbf{a}^*}}(l) \\ &= A_{\mathbf{b},\overleftarrow{\mathbf{a}^*}}(l) - A_{\mathbf{b},\overleftarrow{\mathbf{a}^*}}(l) \\ &= 0 \end{aligned}$$

for every l.

Refer to the proofs of *Theorem 6* and *Theorem 13* from [3]. If $\{\mathbf{a}_1, \mathbf{b}_1\}$ is a complementary pair and $\{\mathbf{a}_2, \mathbf{b}_2\}$ is one of its mates, then both $\{\mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2\}$ and $\{\mathbf{a}_1 \otimes \mathbf{a}_2, \mathbf{b}_1 \otimes \mathbf{b}_2\}$ are complementary pairs. This completes the proof of *Lemma 2.1*.

B. Proof of Theorem 4.3

$$\mathbf{H}_0^{(0)} = \begin{bmatrix} \mathbf{h}_{0,0}^{(0)} \\ \mathbf{h}_{0,1}^{(0)} \end{bmatrix} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}_{2\times 2}$$

and

$$\mathbf{H}_1^{(0)} = \begin{bmatrix} \mathbf{h}_{1,0}^{(0)} \\ \mathbf{h}_{1,1}^{(0)} \end{bmatrix} = \begin{bmatrix} + & - \\ + & + \end{bmatrix}_{2\times 2}$$

It can be verified that $\{\mathbf{h}_{0,0}^{(0)}, \mathbf{h}_{0,1}^{(0)}\}$ and $\{\mathbf{h}_{1,0}^{(0)}, \mathbf{h}_{1,1}^{(0)}\}$ are respectively Golay complementary pairs. Based on *Lemma 2.1*, $\{\mathbf{h}_{0,0}^{(q)}, \mathbf{h}_{0,1}^{(q)}\}$ and $\{\mathbf{h}_{1,0}^{(q)}, \mathbf{h}_{1,1}^{(q)}\}$, $q = 1, 2, 3...$, constructed from (42) are guaranteed to be Golay complementary pairs.

We observe that $f_i(\mathbf{h}_{0,0}^{(0)}) = \mathbf{h}_{1,0}^{(0)}$ and $f_i(\mathbf{h}_{0,1}^{(0)}) = -\mathbf{h}_{1,1}^{(0)}$. Let $f_i(\mathbf{h}_{0,0}^{(q)}) = \mathbf{h}_{1,0}^{(q)}$, $f_i(\mathbf{h}_{0,1}^{(q)}) = -\mathbf{h}_{1,1}^{(q)}$, then,

$$f_i(\mathbf{h}_{0,0}^{(q+1)}) \;=\; f_i(\mathbf{h}_{0,0}^{(q)}\ \overleftarrow{\mathbf{h}_{0,1}^{(q)}}) = f_i(\mathbf{h}_{0,0}^{(q)})\ \overleftarrow{(-f_i(\mathbf{h}_{0,1}^{(q)}))} = \mathbf{h}_{1,0}^{(q)}\ \overleftarrow{\mathbf{h}_{1,1}^{(q)}} = \mathbf{h}_{1,0}^{(q+1)}\ .$$

In a similar way, we have that $f_i(\mathbf{h}_{0,1}^{(q+1)}) = -\mathbf{h}_{1,1}^{(q+1)}$. This ends the proof.

C. Proof of Lemma 6.1

We give the proof for $\lambda_{\mathbf{c}_0}^P \leq \max\left\{\lambda_{\mathbf{s}_0}^P + \lambda_{\mathbf{s}_1}^P, 2\lambda_{\mathbf{s}_0,\mathbf{s}_1}^P\right\}$ and $S_{\mathbf{c}_0}^A \leq S_{\mathbf{s}_0}^A + S_{\mathbf{s}_1}^A + S_{\mathbf{s}_0,\mathbf{s}_1}^A$. Other proofs are similar.

$$\begin{aligned}
\lambda_{\mathbf{c}_0}^P &= \max_l \left\{|P_{\mathbf{c}_0}(l)|, 1 \leq l \leq m-1\right\} \\
&= \max_l \left\{|P_{\mathbf{s}_0}(l) + P_{\mathbf{s}_1}(l)|, 1 \leq l \leq \frac{m}{2} - 1; |P_{\mathbf{s}_0,\mathbf{s}_1}(l) + P_{\mathbf{s}_0,\mathbf{s}_1}(-l-1)|, 0 \leq l \leq \frac{m}{2} - 1\right\} \\
&\leq \max_l \left\{|P_{\mathbf{s}_0}(l)| + |P_{\mathbf{s}_1}(l)|, 1 \leq l \leq \frac{m}{2} - 1; 2|P_{\mathbf{s}_0,\mathbf{s}_1}(l)|, 0 \leq l \leq \frac{m}{2} - 1\right\} \\
&= \max\left\{\lambda_{\mathbf{s}_0}^P + \lambda_{\mathbf{s}_1}^P, 2\lambda_{\mathbf{s}_0,\mathbf{s}_1}^P\right\}
\end{aligned}$$

and

$$\begin{aligned}
S_{\mathbf{c}_0}^A &= \sum_{l=1}^{m-1} |A_{\mathbf{c}_0}(l)| \\
&= \sum_{l=0}^{\frac{m}{2}-1} |A_{\mathbf{s}_0,\mathbf{s}_1}(l) + A_{\mathbf{s}_0,\mathbf{s}_1}(-l-1)| + \sum_{l=1}^{\frac{m}{2}-1} |A_{\mathbf{s}_0}(l) + A_{\mathbf{s}_1}(l)| \\
&\leq \sum_{l=0}^{\frac{m}{2}-1} |A_{\mathbf{s}_0,\mathbf{s}_1}(l)| + \sum_{l=-\frac{m}{2}+1}^{-1} |A_{\mathbf{s}_0,\mathbf{s}_1}(l)| + \sum_{l=1}^{\frac{m}{2}-1} |A_{\mathbf{s}_0}(l)| + \sum_{l=1}^{\frac{m}{2}-1} |A_{\mathbf{s}_1}(l)| \\
&= S_{\mathbf{s}_0}^A + S_{\mathbf{s}_1}^A + S_{\mathbf{s}_0,\mathbf{s}_1}^A
\end{aligned}$$

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