Consider a set of A orthonormal waveforms $\{\varphi_\kappa(\tau), \kappa = 1,\ldots,A\}$, each of which is identically zero outside the time interval [$-\tau$, 0]. Let

$$\overline{\varphi_\kappa(t)} = \frac{1}{A}\sum_{\kappa=1}^{A}\varphi_\kappa(t)$$

be the arithmetic average of these waveforms.

Construct a set of A waveforms $\{\psi_\kappa(t), \kappa = 1,\ldots,A\}$, as follows:

$$\psi_\kappa(t) = \phi_\kappa(t) - \overline{\phi_\kappa(t)}$$

These waveforms are used to construct messaging signals $\{s_i(t)\}$ as follows:

$$s(t) = \sqrt{P_s\tau}\,(\psi_{\kappa 1}(t-\tau) + \psi_{\kappa 2}(t-2\tau) + \psi_{\kappa 3}(t-3\tau) + \ldots + \psi_{\kappa J}(t-J\tau))$$

We can easily recognize that $\{\psi_\kappa(t)\}$ are simplex and therefore they are related via the equation:

$$<\psi_\kappa(t), \psi_i(t)> = \begin{cases} 1-\dfrac{1}{\mathrm{A}}, \kappa = i \\[2ex] -\dfrac{1}{\mathrm{A}}, \kappa \neq i \end{cases}$$

Thus, $||s(t)||^2 = P_s\tau J\left(\dfrac{A-1}{A}\right)$

Because they are simplex, their dimension is N = (A – 1) J.

(a) consider the set of all distinct waveforms in the form of s(t). How many waveforms are there in the set?

A^J

Are any of these waveforms orthogonal?

$$<s_i(t), s_j(t)> = P_s\tau \sum_{n=1}^{J} <\psi_{\kappa in}(t-n\tau), \psi_{\kappa jn}(t-n\tau)> = P_s\tau B\left(1-\frac{1}{A}\right) - \frac{(J-B)}{A}$$

$s_i(t)$ and $s_j(t)$ are orthogonal if $\langle s_i(t), s_j(t)\rangle = 0$ or $B\left(1-\frac{1}{A}\right)-\frac{(J-B)}{A} = 0 \Rightarrow B = \frac{J}{A}$ i.e.

$s_i(t)$ and $s_j(t)$ are orthogonal if they share the same $\{\psi_\kappa(t)\}$ waveforms in $B = \frac{J}{A}$ locations.

Clearly, if J is not an integral multiple of A, none of the $s_i(t)$ waveforms can be orthogonal.

What is the dimensionality N of this ensemble?
$(A - 1)J$

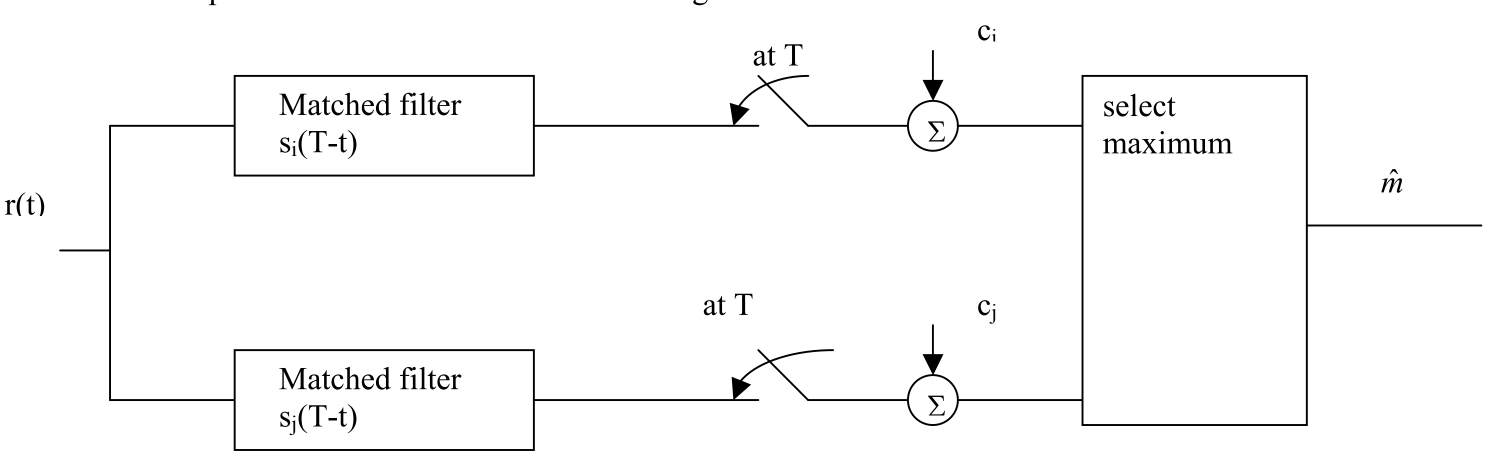
(b) consider the ensemble comprising all waveforms of (a), to each of which is assigned equal probability. Pick two waveforms, independently, at random, from this ensemble. What is the smallest attainable probability of error if these two waveforms are used as the signals in communicating one of two equally likely message over an additive Gaussian channel with power spectral density No/2.

$P[\varepsilon] = Q\left(\frac{d}{\sqrt{2N_o}}\right)$, where $d^2 = \|s(t)_i - s(t)_j\|^2 = 2\|s(t)_i\|^2 - 2\langle s_i(t), s_j(t)\rangle = 2P_s\tau J\left(\frac{A-1}{A}\right) - 2P_s\tau\left(B-\frac{J}{A}\right) = 2P_s\tau h$, where $h = J - B$, the locations that the two waveforms $s_i(t)$ and $s_j(t)$ differ.

Thus, $P[\varepsilon] = Q\left(\sqrt{\frac{P_s\tau h}{N_o}}\right)$

Denote the received signal as r(t) and describe the receiver which attains this smallest pair wise error probability.

The optimum receiver is illustrated in the figure below.



(c) what is the average $\overline{P_2(\varepsilon)}$ of the error probability of (b) over the ensemble in (a)?

Let's name P(h) the probability the $s_i(t)$ and $s_j(t)$ to differ in h places. That is

$$P(h) = \binom{J}{h}\left(\frac{1}{A}\right)^{J-h}\left(1-\frac{1}{A}\right)^{h}, h = 0 \ldots J$$

$$\overline{P_2(\varepsilon)} = \sum_{h=0}^{J} P(h) Q\left(\sqrt{\frac{P_s \tau h}{N_o}}\right) \le A^{-J} \sum_{h=0}^{J} \binom{J}{h} (A-1)^{h} \exp\left(-\frac{P_s \tau h}{2N_o}\right)$$

Compute the exponential upper bound R_0 on the average pairwise error probability.

After applying the relation

$$\sum_{h=0}^{J} \binom{J}{h} a^{h} = [1+a]^{h}, \text{ with } a = (A-1)\exp\left(-\frac{P_s \tau}{2N_o}\right)$$

$$\overline{P_2(\varepsilon)} \le A^{-J}\left[1+(A-1)\exp\left(-\frac{P_s \tau}{2N_o}\right)\right]^{J} = 2^{\left\{-J \log_2 \frac{A}{\left[1+(A-1)\exp\left(-\frac{Ps\tau}{2No}\right)\right]}\right\}}$$

Since N = J(A-1) or J = N / (A –1)

The above relation can be written as:

$$\overline{P_2(\varepsilon)} \le 2^{\left\{-N \frac{1}{A-1} \log_2 \frac{A}{\left[1+(A-1)\exp\left(-\frac{Ps\tau}{2No}\right)\right]}\right\}}$$

This relation shows that R_0 is $\frac{1}{A-1} \log_2 \frac{A}{\left[1+(A-1)\exp\left(-\frac{P_s \tau}{2N_o}\right)\right]}$

(d) express the upper bound on the average probability of error, $\overline{P(\varepsilon)}$, for $M = 2^{NR_N}$ equally likely messages (R_N being number of bits per signal dimension) in terms of R_N and R_0 when M signals are drawn independently at random from the ensemble in(b),

Using Union Bound:

$$\overline{P(\varepsilon)} \le M 2^{-NR_0} = 2^{NR_N} 2^{-NR_0} = 2^{-N(R_0 - R_N)}$$

(e) express the energy per bit E_b in terms of parameters introduced earlier and compare the minimum energy E_b/N_0 for which the bound in (d) still allows for error free transmission with the corresponding minimum energy E_b/N_0 for the binary antipodal signaling.

Number of Bits = K = NR_N

$$E_b = E_s/K = \frac{Ps\,\tau J\left(\frac{A-1}{A}\right)}{K} = \frac{Ps\,\tau}{AR_N}$$

For error free transmission: $R_0 - R_N > 0$ or

$$N\left(R_0 - R_N\right) > 0 \Rightarrow$$

$$NR_N\left\{\left[\frac{1}{(A-1)R_N}\log_2\frac{A}{\left[1+(A-1)\exp\left(-\frac{P_s\tau}{2N_o}\right)\right]}\right]-1\right\}>0$$

Let's set $x = \frac{P_s\tau}{2N_o}$. The above relation can be written as

$$K\left\{\left[\frac{1}{(A-1)R_N}\log_2\frac{A}{\left[1+(A-1)\exp(-x)\right]}\right]-1\right\}>0\Rightarrow$$

$$\frac{1}{(A-1)R_N}\log_2\frac{A}{\left[1+(A-1)\exp(-x)\right]}>1$$

For minimum energy to signal ratio, we consider x->0.

Therefore we have the relation

$$\lim_{x->0}\left[\frac{1}{(A-1)R_N}\log_2\frac{A}{\left[1+(A-1)\exp(-x)\right]}\right]>1\Rightarrow$$

$$\lim_{x->0}\left[\frac{1}{\frac{2N_0}{E_b}x}\log_2\frac{A}{\left[1+(A-1)\exp(-x)\right]}\right]>1\Rightarrow$$

$$\frac{1}{\ln 2}\lim_{x->0}\left[\frac{1}{\frac{2N_0}{E_b}}\frac{\left[1+(\mathrm{A}-1)\exp(-x)\right]\mathrm{A}}{\mathrm{A}\left[1+(\mathrm{A}-1)\exp(-x)\right]^2}\right]>1\Rightarrow$$

$$\frac{1}{2\ln 2}\frac{E_b}{N_0}>1$$

In the above series of equation we have used the relation:

$$E_b=\frac{\mathrm{Ps}\,\tau}{\mathrm{A}R_N}=\frac{2N_o x}{\mathrm{A}R_N}\Rightarrow \mathrm{A}R_N=x\frac{2N_o}{E_b} \text{ and } (\mathrm{A}-1)R_N=x\frac{2N_o}{E_b}$$

and we have used l’Hôpital's Rule regarding limits.

For block orthogonal signaling eq. 5.15b reads $\left(\frac{E_b}{N_o}\right)_{\min}^{PPM}=2\ln 2$

We may notice that the energy requirements are the same.