Optimum Multiuser Detection is Tractable for Synchronous CDMA Systems Using M-sequences

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Abstract— **The optimum multiuser detection problem was shown to be NP-hard i.e., its computational complexity increases exponentially with the number of users [1, 2]. In this paper, we show that the optimum multiuser detection problem for a synchronous CDMA system is equivalent to the minimum capacity cut problem in a related network and propose an optimum multiuser detection algorithm with polynomial computational complexity for a certain class of signature sequences. The minimum cut problem is solvable in polynomial time if the capacities of the links not incident to source and sink are nonnegative. This condition in the optimum detection problem is equivalent to all cross correlations between the signature sequences of the users being negative. One example of such set of signature sequences is obtained when shifted versions of the the maximal length sequences (or m-sequences) are used. In this case the cross correlation between users i and j is given as, $\Gamma_{ij} = -\frac{1}{G}$ for all i, j, where G is the processing gain.**

Keywords— **Optimum multiuser detection, CDMA**

I. Introduction

In CDMA systems users are assigned unique signature waveforms which they use to modulate their information bits. Let the signature sequence of the ith user be $s_i(t)$ for $t \in [0, T]$ where T is the bit duration. The received signal for a synchronous CDMA system with BPSK modulation is given by

$$r(t) = \sum_{i=1}^{N} A_i a_i s_i(t) + n(t) \tag{1}$$

where A_i and a_i are received amplitude and the transmitted bit (± 1 equiprobably) of the ith user and $n(t)$ is the additive white Gaussian noise (AWGN) process with power spectral density σ^2. The received signal vector at the output of the conventional receivers is given by

$$\boldsymbol{y} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{a} + \boldsymbol{n} \tag{2}$$

The vector $\boldsymbol{y}$ is a sufficient statistics for the multiuser detection problem. In (2), $\boldsymbol{\Gamma}$ is a nonnegative definite matrix where Γ_{ij} is $\int_0^T s_i(t)s_j(t)dt$, $\boldsymbol{\Lambda}$ is a diagonal matrix containing the received amplitudes of the users with $\Lambda_{ii} = A_i$, $\boldsymbol{a}$ is the vector containing the information bits of the users and $\boldsymbol{n}$ is a Gaussian random vector with auto covariance matrix $E[\boldsymbol{n}\boldsymbol{n}^\top] = \sigma^2\boldsymbol{\Gamma}$.

The aim of the multiuser detection is to recover the information bits transmitted by the users in this multi access environment. Optimum multiuser detection [1] is based on the maximum likelihood criteria. The optimum multiuser detector chooses $\boldsymbol{a}^*$ as the transmitted bit vector if for $\boldsymbol{a} = \boldsymbol{a}^*$ the conditional probability density of $\boldsymbol{y}$ given $\boldsymbol{a}$ is maximized. Denoting the probability density function of $\boldsymbol{n}$ by $f_{\boldsymbol{n}}(.)$, the optimum detection problem is given as

$$\begin{aligned} \boldsymbol{a}^* &= \arg \max_{\boldsymbol{a} \in \{-1,1\}^N} f_{\boldsymbol{n}}(\boldsymbol{y} - \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{a}) \\ &= \arg \min_{\boldsymbol{a} \in \{-1,1\}^N} \boldsymbol{a}^\top \boldsymbol{R}\boldsymbol{a} - 2\boldsymbol{a}^\top \boldsymbol{\Lambda}\boldsymbol{y} \end{aligned} \tag{3}$$

where $\boldsymbol{R} = \boldsymbol{\Lambda}\boldsymbol{\Gamma}\boldsymbol{\Lambda}$ with $R_{ij} = A_i A_j \Gamma_{ij}$. We can convert (3) to a 0-1 programming problem by introducing a vector $\boldsymbol{b}$ where $\boldsymbol{b} = (\boldsymbol{a} + \boldsymbol{u})/2$ and $\boldsymbol{u}$ is an N dimensional vector of all ones, $\boldsymbol{u} = [1\ 1\ 1 \dots 1]^\top$ as

$$\boldsymbol{b}^* = \arg \min_{\boldsymbol{b} \in \{0,1\}^N} \boldsymbol{b}^\top \boldsymbol{R}\boldsymbol{b} - \boldsymbol{b}^\top \tilde{\boldsymbol{y}} \tag{4}$$

where $\tilde{\boldsymbol{y}} = \boldsymbol{R}\boldsymbol{u} + \boldsymbol{\Lambda}\boldsymbol{y}$. Note that the solutions of (3) and(4) are related by the one-to-one relationship $a_i^* = 2b_i^* - 1$.

II. Network Preliminaries

Consider a network $G = [V, A]$ with vertices $V = \{0, 1, \dots, N+1\}$ and arcs A. For any two vertices i and j in G, c_{ij} denotes the capacity of the arc connecting (i, j). Let the nodes 0 and $N+1$ represent the source and the sink respectively. A cut separating 0 and $N+1$ is a partition of the nodes $(\mathcal{S}, \bar{\mathcal{S}})$ where $0 \in \mathcal{S}$, $N+1 \in \bar{\mathcal{S}}$, $\mathcal{S} \cup \bar{\mathcal{S}} = V$ and $\mathcal{S} \cap \bar{\mathcal{S}} = \emptyset$. The capacity of the cut $(\mathcal{S}, \bar{\mathcal{S}})$ is given by [3]

$$C(\mathcal{S}, \bar{\mathcal{S}}) = \sum_{i \in \mathcal{S}} \sum_{j \in \bar{\mathcal{S}}} c_{ij} \tag{5}$$

The minimum cut separating nodes 0 and $N+1$ is defined to be the cut separating nodes 0 and $N+1$ and having the minimum capacity.

In [4] it was shown that any cut separating nodes 0 and $N+1$ can be represented by a vector $(1, b_1, b_2, \dots, b_N, 0)$ where $b_i \in \{0, 1\}$ for $i = 1, \dots, N$ is an indication for membership in $\mathcal{S}$. That is $\mathcal{S} = \{i \mid b_i = 1\}$ and $\bar{\mathcal{S}} = \{i \mid b_i = 0\}$. It was also shown in [4] that the capacity of the cut $(\mathcal{S}, \bar{\mathcal{S}})$ is given by

$$C(\boldsymbol{b}) = \sum_{i=0}^{N+1} \sum_{j=0}^{N+1} c_{ij} b_i (1 - b_j) \tag{6}$$

where $b_0 = 1$ and $b_{N+1} = 0$. Substituting the values of b_0 and b_{N+1} in (6) and using the fact that $b_i^2 = b_i$ yields

$$C(\boldsymbol{b}) = \sum_{j=1}^{N+1} c_{oj} - \sum_{i=1}^{N} \left(c_{0i} - \sum_{j=1}^{N+1} c_{ij} \right) b_i - \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} b_i b_j \tag{7}$$

Similar to [5], we show the equivalence of the optimum multiuser detection problem to a minimum cut problem in a related network and identify the conditions under which it can be solved in polynomial time in the following two sections.

III. Relation between Minimum Cut and Optimum Multiuser Detection Problems

Let us define the objective function of the 0-1 programming version the optimum multiuser detection problem given in (4) to be $F(\boldsymbol{b})$. Then,

$$F(\boldsymbol{b}) = -\sum_{i=1}^{N} \tilde{y}_i b_i + \sum_{i=1}^{N}\sum_{j=1}^{N} R_{ij} b_i b_j \qquad (8)$$

Comparing (7) and (8) we observe that if the capacities c_{ij} are chosen properly, the cost function of (8) in N variables can be represented as the capacity of a cut in an associated network with $(N+2)$ nodes. Thus, a network G with arc capacities c_{ij} satisfying:

$$\begin{array}{lll} \text{(a)} & c_{ij} = -R_{ij} & \text{for} \quad i,j = 1,\ldots,N \\ \text{(b)} & c_{0i} - \sum_{j=1}^{N+1} c_{ij} = \tilde{y}_i & \text{for} \quad i = 1,\ldots,N \\ \text{(c)} & \sum_{j=0}^{N+1} c_{0j} = 0 & \end{array} \qquad (9)$$

has $C(\boldsymbol{b}) = F(\boldsymbol{b})$ for all $\boldsymbol{b}$ such that $b_i \in \{0,1\}$ for $i = 1,\ldots,N$. Given the parameters of the quadratic 0-1 programming problem $\tilde{\boldsymbol{y}}$ and $\boldsymbol{R}$, construction of the corresponding network satisfying above three conditions is not unique. The condition (a) of (9) fixes the capacities of all arcs not incident to the source and the sink. Since there are more arcs than the conditions to be satisfied given in (9)-(b) and (c), some of the arc capacities can be assigned arbitrary numbers while conserving the connectivity of the overall network. One such assignment equates $c_{N+1,j}$ to zero for $j = 0,\ldots,N$ and c_{0j} to arbitrary positive numbers for $j = 1,\ldots,N$. Then rest of the $(N+1)$ arc capacities can be determined uniquely from conditions (9)-(b) and (c). From (9)-(b)

$$c_{i,N+1} = -\tilde{y}_i - \sum_{j=1}^{N} c_{ij} + c_{0i} = -A_i y_i + c_{0i} \qquad (10)$$

where we used $\tilde{y}_i = \sum_{j=1}^{N} R_{ij} + A_i y_i$ from the definition of $\tilde{\boldsymbol{y}}$ and (9)-(a) which implies $\sum_{j=1}^{N} R_{ij} = -\sum_{j=1}^{N} c_{ij}$. The only remaining capacity, $c_{0,N+1}$, can be found from (9)-(c). The following rules construct the related network for the optimum multiuser detection problem:

(i) Set $c_{ij} = -R_{ij}$ for $i,j = 1,\ldots,N$

(ii) Set $c_{i0} = 0$ and assign arbitrary positive numbers to c_{0i} for $i = 1,\ldots,N$ and $c_{0,N+1} = -\sum_{j=1}^{N} c_{0j}$

(iii) Set $c_{i,N+1} = -A_i y_i + c_{0i}$ for $i = 1,\ldots,N$

In the following subsection we demonstrate the equivalence of the optimum multiuser detection problem and the minimum cut problem in the related network obtained by the rules given above for the case of two users.

A. A Simple Example: Two-User Case

Consider a two user system with the cross correlation between the signature sequences of the users $\Gamma_{12} = \Gamma_{21} = -\rho$, for some $0 \leq \rho \leq 1$. Then $R_{12} = R_{21} = -A_1 A_2 \rho$. Since the optimum multiuser detection problem (3) is insensitive to the diagonal elements of $\boldsymbol{R}$ we take $R_{11} = R_{22} = 0$. The optimum multiuser detection problem of (4) is to minimize $F(b_1, b_2)$ for $b_1, b_2 \in \{0,1\}$ where $F(b_1, b_2)$ is given as

$$-2A_1A_2\rho b_1 b_2 + (A_1A_2\rho - A_1 y_1) b_1 + (A_1A_2\rho - A_2 y_2) b_2$$

For all possible transmitted bits the cost function $F(b_1, b_2)$ have the following values:

$$\begin{array}{ll} F(0,0) = 0 & F(0,1) = A_1A_2\rho - A_2 y_2 \\ F(1,0) = A_1A_2\rho - A_1 y_1 & F(1,1) = -A_1 y_1 - A_2 y_2 \end{array} \qquad (11)$$

The optimum detection rule chooses that realization of (b_1, b_2) for which the cost function $F(b_1, b_2)$ is minimum. Clearly, the minimum value depends on cross correlation value (ρ), received amplitudes of the users (A_1, A_2) and the matched filter outputs (y_1, y_2). The graph corresponding to this multiuser detection problem is shown in Figure 1. The rules given above are used to construct the graph:

(i) $c_{12} = c_{21} = A_1 A_2 \rho$,

(ii) $c_{10} = c_{20} = c_{30} = 0$, $c_{01} = \alpha$, $c_{02} = \beta$, $c_{03} = -(\alpha+\beta)$,

(iii) $c_{13} = \alpha - A_1 y_1$, $c_{23} = \beta - A_2 y_2$.

The capacities of the cuts corresponding to 4 possible cases for (b_1, b_2), $C(b_1, b_2)$, are calculated from Figure 1 to be:

$$\begin{array}{ll} \text{CUT}_1: & C(0,0) = 0 \\ \text{CUT}_2: & C(0,1) = A_1A_2\rho - A_2 y_2 \\ \text{CUT}_3: & C(1,0) = A_1A_2\rho - A_1 y_1 \\ \text{CUT}_4: & C(1,1) = -A_1 y_1 - A_2 y_2 \end{array}$$

Note the equivalence of the values of the cost function in (11) and the capacities of the cuts calculated above.

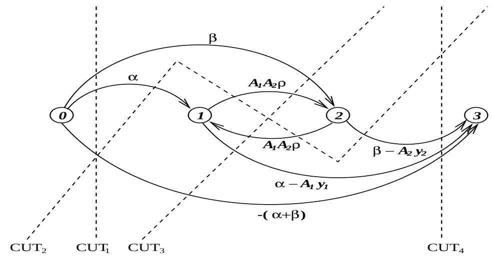

Fig. 1. Graph associated with the optimum MUD problem for $N = 2$.

IV. Optimum Multiuser Detectors with Polynomial Complexity

The arguments in Section III show that the optimum multiuser detection problem is equivalent to finding a minimum capacity cut between the source and the sink nodes of a related network. For a general optimum multiuser detection problem the capacities of the arcs in the related network can be both positive and negative depending on the cross correlations between the signature sequences. Unfortunately, in this general case the minimum cut problem is an NP-hard problem as well [5]. However, in the special case when the capacities of the arcs not incident to both source and sink are nonnegative, the minimum cut problem is solvable in polynomial time. The minimum cut problem is the dual of the maximum flow problem for which there are many polynomial time algorithms [3]. For example the computational complexity of FIFO preflow-push algorithm [3] is $O(N^3)$ where N is the number of nodes of the network. This implies that the computational complexity of the optimum multiuser detection problem is $O(N^3)$ where N is the number of active users, because if the dimensionality of the optimum multiuser detection problem is N then the number of nodes in the associated network is $(N+2)$.

From rule (i) we observe that the capacities of the arcs not incident to source and sink are guaranteed to be nonnegative only if $R_{ij} \leq 0$ for all $i \neq j$. Note that $R_{ij} \leq 0$ does not apply to the diagonal elements of $\boldsymbol{R}$, since the optimum multiuser detection problem of (3) is insensitive to the values of the diagonal elements of $\boldsymbol{R}$, because $a_i^2 = 1$ for $a_i \in \{-1, 1\}$. Moreover, since $R_{ij} = A_i A_j \Gamma_{ij}$ the condition that $R_{ij} \leq 0$ for $i \neq j$ directly translates to the condition that $\Gamma_{ij} \leq 0$ for $i \neq j$. Thus, the optimum multiuser detection problem has a polynomial computational complexity only if the signature sequences of the users are chosen such that all of the cross correlations are negative. This fact suggests appropriate design of the signature sequence set to be used by the users. One such set of signature sequences is obtained when shifted versions of the *m-sequences* are used (see [6] for example). The auto correlation of m-sequences is equal to $-1/G$ for all nonzero shifts, where G is the length of the sequence. Therefore, if shifted versions of m-sequences are used as the signature sequences, the cross correlation between any two users becomes $\Gamma_{ij} = -\frac{1}{G}$ for $i \neq j$ where G is the processing gain.

The steps of polynomial time optimum multiuser detection algorithm for N users with all negative cross correlations are: (1) construct the related network with $(N+2)$ nodes, (2) solve the minimum cut problem in this network using a polynomial time algorithm such as FIFO preflow-push [3], (3) using the membership function declare $\boldsymbol{b}^*$. Note that while constructing the related network only a linear number of arc capacities (the ones that depend on $\boldsymbol{y}$) needs to be changed from bit to bit, because others depend only on $\boldsymbol{R}$ which is fixed once the signature sequences and the received amplitudes of the users are fixed.

V. Conclusion

In this paper we showed that if a set of signature sequences where all cross correlations are negative is used, then the optimum multiuser detection problem is solvable in $O(N^3)$ time.

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