16:330:541 **Stochastic Signals and Systems** **First Exam**

October 26, 1998

Solution

You have 100 minutes to complete this exam. The point value of each part is indicated. Please read the exam carefully and ask the instructor if you have any questions.

1. A certain device contains n circuits and it works if one or more of those circuits work. Each circuit fails with probability q, independent of any other circuit. A working device, even if it contains one or more broken circuits, can be sold for \$50 but a broken device must be discarded.

   (a) *10 points* Suppose you test devices until you find a working device. What is the PMF of K, the number of devices you test?

   A device works with probability

   $$p = P[W] = 1 \Leftrightarrow q^n$$

   Each device test is a Bernoulli trial so K is the number of trials until the first success. Hence K has the geometric PMF

   $$P_K(k) = \begin{cases} (1 \Leftrightarrow p)^{k-1} p & k = 1, 2, \ldots \\ 0 & \textit{otherwise} \end{cases} = \begin{cases} q^{n(k-1)}(1 \Leftrightarrow q^n) & k = 1, 2, \ldots \\ 0 & \textit{otherwise} \end{cases}$$

   (b) *10 points* If each circuit costs \$5, what is the expected profit $E[R]$ per device?

   Each working device is worth \$50 but each nonworking device is worth nothing. Thus the average profit per device is

   $$E[R] = 50P[W] \Leftrightarrow 5n = 50(1 \Leftrightarrow q^n) \Leftrightarrow 5n$$

   (c) *10 points* Suppose $n = 5$. What is the PMF of D, the number of working circuits in a device? Given the event W that the device works, find the conditional PMF of D given the event W.

   Whether each of the 5 circuits in a device works is a Bernoulli trial with success probability $1 \Leftrightarrow q$. Thus D, the number of successes in 5 trials, has the binomial PMF

   $$P_D(d) = \begin{cases} \binom{5}{d}(1 \Leftrightarrow q)^d q^{5-d} & d = 0, 1, \ldots, 5 \\ 0 & \textit{otherwise} \end{cases}$$

   The event W is identical to the event $D > 0$ which has probability

   $$P[W] = P[D > 0] = 1 \Leftrightarrow P_D(0) = 1 \Leftrightarrow q^5$$

   The conditional PMF of D given W is

   $$P_{D|W}(d) = \begin{cases} \frac{P_D(d)}{P[W]} & d \in W \\ 0 & \textit{otherwise} \end{cases} = \begin{cases} \frac{\binom{5}{d}(1-q)^d q^{5-d}}{1-q^5} & d = 1, \ldots, 5 \\ 0 & \textit{otherwise} \end{cases}$$

(d) *10 points* Ultrareliable circuits cost \$10 each but fail with probability $q/2$. Suppose $n = 2$ and you have the option of substituting zero, one or two ultrareliable circuits for the ordinary circuits. Let $E[R_i]$ denote your expected profit using i ultrareliable circuits. For what values of q is it best to use exactly 1 ultrareliable circuit?

Let W_i denote the event that a device with i ultrareliable circuits works. We observe that

$$P[W_0] = 1 \Leftrightarrow q^2 \qquad P[W_1] = 1 \Leftrightarrow q(q/2) \qquad P[W_2] = 1 \Leftrightarrow (q/2)^2$$

Let R_i denote the profit on a device with i ultrareliable circuits. A working device is worth \$50. With i ultrareliable circuits, the cost of a device is $5(2 \Leftrightarrow i) + 10i = 10 + 5i$. This implies

$$E[R_i] = 50P[W_i] \Leftrightarrow (10+5i) = \begin{cases} 50(1 \Leftrightarrow q^2) \Leftrightarrow 10 & i = 0 \\ 50(1 \Leftrightarrow q^2/2) \Leftrightarrow 15 & i = 1 \\ 50(1 \Leftrightarrow q^2/4) \Leftrightarrow 20 & i = 2 \end{cases}$$

Its easy to verify that

$$E[R_1] > E[R_0] \Leftrightarrow q > \sqrt{1/5} \qquad E[R_1] > E[R_2] \Leftrightarrow q < \sqrt{2/5}$$

Thus 1 ultrareliable circuit is best iff $\sqrt{1/5} < q < \sqrt{2/5}$.

(e) *10 points* Let M equal the number of circuits tested in order to find enough good circuits for 100 devices. What is the PMF of M?

As noted during the exam, this part is a little confusing. In fact, you need only one good circuit in each device and so you need only 100 good circuits for 100 devices. Thus, M is the number of tests needed to identify 100 good circuits and has a Pascal PMF. Since a circuit is good with probability $1 \Leftrightarrow q$, the PMF of M is

$$P_M(m) = \begin{cases} \binom{m-1}{99}(1 \Leftrightarrow q)^{99} q^{m-99} & m = 100, 101, \ldots \\ 0 & \text{otherwise} \end{cases}$$

2. *50 points* Random variable X is Gaussian with zero mean and unit variance. Given $X = x$, Y is a Gaussian random variable with mean $10x$ and variance 1. Note that some parts of this problem are easiest using conditional expectations. Find $E[Y]$, $E[Y^2]$, $\sigma_{X,Y}$, and $f_{X,Y}(x,y)$. Do X and Y have a bivariate Gaussian PDF? Justify your answer carefully.

In the problem statement, we learn that the conditional PDF of Y given $X = x$ is Gaussian and the following two facts:

$$E[Y|X = x] = 10x \qquad \mathrm{Var}[Y|X = x] = 1$$

The second fact is misleading. It appears that if the conditional variance is always 1, then perhaps $\mathrm{Var}[Y]$ should always be 1. We will see that's not the case. First we observe that since $E[Y|X = x] = 10x$ then $E[Y|X] = 10X$. Thus

$$E[Y] = E[E[Y|X]] = E[10X] = 10E[X] = 0$$

To find $E[Y^2]$, we observe that

$$E[Y^2|X = x] = \mathrm{Var}[Y|X = x] + (E[Y|X = x])^2 = 1 + (10x)^2$$

Thus, $E\left[Y^2|X\right] = 1 + 100X^2$. *Since* X *has zero mean,* $E\left[X^2\right] = \mathrm{Var}[X] = 1$, *and*

$$E\left[Y^2\right] = E\left[E\left[Y^2|X\right]\right] = E\left[1 + 100X^2\right] = 101$$

Thus we see that $\mathrm{Var}[Y] = E\left[Y^2\right] = 101$. *That is,* $\mathrm{Var}[Y] \neq 1$ *even though* $\mathrm{Var}[Y|X = x] = 1$ *for all* x. *Since the conditional mean of* Y *depends strongly on* X, *the variation of* X *results in a large variance for* Y.

Since both X *and* Y *are zero mean,* $\mathrm{Cov}[X,Y] = E[XY]$. *Once again, we use conditional expectations:*

$$E[XY|X = x] = E[xY|X = x] = xE[Y|X = x] = 10x^2$$

Thus $E[XY|X] = 10X^2$ *and*

$$E[XY] = E[E[XY|X]] = E\left[10X^2\right] = 10E\left[X^2\right] = 10$$

From the problem statement, we learned that

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}}e^{-(y-10x)^2/2}$$

The joint PDF of X *and* Y *is*

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x) = \frac{1}{2\pi}e^{-[x^2+(y-10x)^2]/2} = \frac{1}{2\pi}e^{-(101x^2-20xy+y^2)/2}$$

When $E[X] = E[Y] = 0$, *the bivariate Gaussian PDF is*

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1 \Leftrightarrow \rho^2}}e^{-\left[\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right]/2(1-\rho^2)}$$

In our specific problem, we know that $\sigma_X = 1$ *and* $\sigma_Y = \sqrt{101}$. *This implies that the correlation coefficient of* X *and* Y *is*

$$\rho = \frac{\mathrm{Cov}[X,Y]}{\sigma_X\sigma_Y} = \frac{10}{\sqrt{101}} = \sqrt{\frac{100}{101}}$$

Its easy to verify that using the above value of ρ *and* $\sigma_X = 1$, $\sigma_Y = \sqrt{101}$ *in the bivariate Gaussian PDF will give the joint PDF* $f_{X,Y}(x,y)$. *Thus* X *and* Y *are bivariate Gaussian random variables.*

3. *50 points* $X_1, X_2, \ldots, X_n$ are iid random variables, each uniformly distributed over $[0,1]$.

   (a) *10 points* Find the joint PDF $f_{X_1,X_2}(x_1,x_2)$.
   This is really easy. Each X_i *has PDF*

$$f_{X_i}(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

   Since X_1 *and* X_2 *are independent, the joint PDF is*

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2) = \begin{cases} 1 & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

(b) *10 points* Find the mean and variance of $W = X_1 + \cdots + X_n$.
Its always true that the expectation of the sum is the sum of the expectations:

$$E[W] = E[X_1] + \cdots + E[X_n] = nE[X] = n/2$$

Note that you should recall that if X has a uniform PDF on $[0,1]$, then $\text{Var}[X] = 1/12$. *Since the X_i are independent, the variance of the sums is the sum of the variances and*

$$\text{Var}[W] = \text{Var}[X_1] + \cdots + \text{Var}[X_n] = n\,\text{Var}[X] = n/12$$

(c) *10 points* Find the probability $P[A] = P[X_1 \leq X_2 \leq \cdots \leq X_n]$.
One way to find $P[A]$ is to use a symmetry argument. Let $x_1, \ldots, x_n$ denote the sample values of $X_1, \ldots, X_n$. Since the X_i are iid, it would seem likely that all $n!$ orderings (from smallest to largest) of $x_1, \ldots, x_n$ are equally likely. Since $x_1, \ldots, x_n$ is exactly one of these orderings, we should have $P[A] = 1/n!$. Note that because the X_i are continuous, we don't need to consider tricky cases where $X_i = X_j$. If you don't trust your intuition, it is also possible to solve this problem with calculus:

$$\begin{aligned}
P[A] &= \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \int_0^{x_{n-2}} dx_{n-2} \cdots \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \\
&= \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \int_0^{x_{n-2}} dx_{n-2} \cdots \int_0^{x_3} x_2 dx_2 \\
&= \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \int_0^{x_{n-2}} dx_{n-2} \cdots \int_0^{x_4} \frac{x_3^2}{2} dx_3 \\
&\vdots \\
&= \int_0^1 \frac{x_n^{n-1}}{(n \Leftrightarrow 1)!} dx_n = \frac{1}{n!}
\end{aligned}$$

(d) *20 points* Find the PDF of $Y = X_3 X_4$.
Just as in part (a), the joint PDF of X_3 and X_4 is

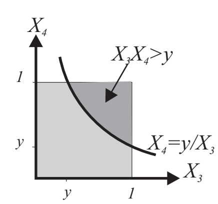
$$f_{X_3,X_4}(x_3,x_4) = f_{X_3}(x_3)\, f_{X_4}(x_4) = \begin{cases} 1 & 0 \leq x_3 \leq 1, 0 \leq x_4 \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

To find the PDF of Y, we first find the CDF. Since $0 \leq Y \leq 1$, we know that $F_Y(y) = 0$ for $y < 0$ and $F_Y(y) = 1$ for $y \geq 1$. For $0 \leq y \leq 1$, we observe that

$$F_Y(y) = P[Y \leq y] = P[X_3 X_4 \leq y] = 1 \Leftrightarrow P[X_4 > y/X_3]$$

At this point, it helps to refer to the sketch below. For $0 \leq y \leq 1$

$$\begin{aligned}
F_Y(y) &= 1 \Leftrightarrow \int_y^1 \left(\int_{y/x_3}^1 dx_4 \right) dx_3 \\
&= 1 \Leftrightarrow \int_y^1 (1 \Leftrightarrow y/x_3)\, dx_3 \\
&= y \Leftrightarrow y \ln y
\end{aligned}$$



By taking the derivative, we find the PDF

$$f_Y(y) = \begin{cases} \Leftrightarrow \ln y & 0 < y \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$