Solutions

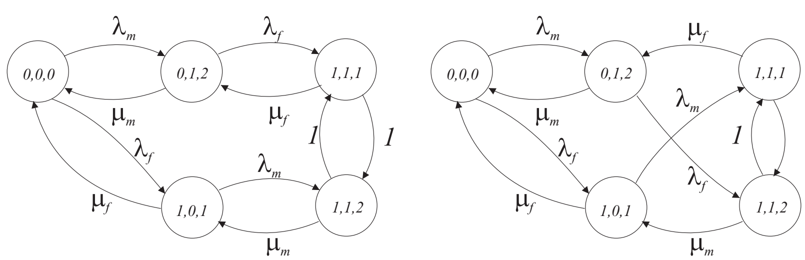
December 21, 1998

You have 180 minutes to answer the following problems. The point value of each part is indicated. You may use two sides of two sheets of notes. Please make sure that you have included your name, ID number and signature in each book used. Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. *60 points* Farmers and Mechanics Bank is a very peculiar bank that serves both kinds of customers: farmers and mechanics. Farmers and mechanics arrive as independent Poisson processes of rate λ_f and λ_m customers/minute. However, if there already is one farmer in the bank, all other farmers are blocked from entering. Similarly, no more than one mechanic can enter the bank at a time. Each blocked customer immediately departs, never to be seen again. Each farmer or mechanic that enters the banks has an independent exponential service requirement with mean $1/\mu_f$ minutes for a farmer or $1/\mu_m$ minutes for a mechanic. The bank has only a single teller. If there is one customer in the bank, that customer is served. With both a farmer and a mechanic in the bank, the teller switches back and forth between the two customers, spending an independent exponential time with an average of 1 minute (assuming the customer doesn't finish service and leave) with each customer before switching.

   (a) *15 points* Let the 3-tuple fms denote the state of the system where f is the number of farmers and m is the number of mechanics in the bank, and s is the type of customer currently in service served. That is, $s = 0$ if the system is empty, $s = 1$ if a farmer is being served and $s = 2$ is a mechanic is being served. Draw a continuous time Markov chain for the system. How many communicating classes does this chain have?

   There is a slight ambiguity in the problem statement. In particular, when a farmer is in the system and a mechanic arrives, does the server switch to the mechanic? Similarly, when a mechanic is in service and a farmer arrives, does the server switch? The two possible chains are



Server switches *Server does not switch*

   In either case, the chain has only one communicating class.

   (b) *15 points* Find the stationary probabilities p_{fms} when $\lambda_f = 1 = \lambda_m = \mu_f = \mu_m = 1$. Hint: perhaps for neighboring states, fms and $f'm's'$, $p_{fms}p_{fms,f'm's'} = p_{f'm's'}p_{f'm's',fms}$?

   The hint was a little misleading because the notation was wrong. The correct notation was $p_{fms}q_{fms,f'm's'} = p_{f'm's'}q_{f'm's',fms}$. Worse than that, the hint was no help for the nonswitching case.

   For the switching case, finding the stationary probabilities is easy. Let $\rho_f = \lambda_f/\mu_f$ and

$\rho_m = \lambda_m/\mu_m$. *Using the hint we try to solve*

$$\begin{aligned}
p_{0,0,0}\lambda_f = p_{1,0,1}\mu_f &\quad\Rightarrow\quad p_{1,0,1} = \rho_f p_{0,0,0} \\
p_{1,0,1}\lambda_m = p_{1,1,2}\mu_m &\quad\Rightarrow\quad p_{1,1,2} = \rho_m p_{1,0,1} = \rho_m \rho_f p_{0,0,0} \\
p_{0,0,0}\lambda_m = p_{0,1,2}\mu_m &\quad\Rightarrow\quad p_{0,1,2} = \rho_m p_{0,0,0} \\
p_{0,1,2}\lambda_f = p_{1,1,1}\mu_f &\quad\Rightarrow\quad p_{1,1,1} = \rho_m p_{0,1,2} = \rho_m \rho_f p_{0,0,0}
\end{aligned}$$

Lastly, we observe that $p_{1,1,1} = p_{1,1,2}$. *Its easy to show that these stationary probabilities satisfy global balance. For example, at state* $0, 1, 2$, *global balance yields*

$$p_{0,1,2}(\lambda_f + \mu_m) = \left(\frac{\lambda_f \lambda_m}{\mu_m} + \lambda_m\right) p_{0,0,0} = \mu_f p_{1,1,1} + \lambda_m p_{0,0,0}$$

To find $p_{0,0,0}$, *we make sure the state probabilities add up to one:*

$$p_{0,0,0} + p_{1,0,1} + p_{0,1,2} + p_{1,1,1} + p_{1,1,2} = p_{0,0,0}\left(1 + \rho_m + \rho_f + 2\rho_m\rho_f\right) = 1$$

Thus,

$$p_{0,0,0} = \frac{1}{1 + \rho_m + \rho_f + 2\rho_m\rho_f}$$

When $\lambda_f = \lambda_m = \mu_f = \mu_m = 1$, *we have* $\rho_f = \rho_m = 1$ *and the stationary probabilities are*

$$p_{0,0,0} = 1/5 \quad p_{1,0,1} = 1/5 \quad p_{0,1,2} = 1/5 \quad p_{1,1,1} = 1/5 \quad p_{1,1,2} = 1/5$$

For the nonswitching case, we can solve the following global balance equations:

$$\begin{aligned}
2p_{0,1,2} &= p_{0,0,0} + p_{1,1,1} \\
2p_{1,0,1} &= p_{0,0,0} + p_{1,1,2} \\
2p_{1,1,1} &= p_{1,0,1} + p_{1,1,2} \\
2p_{1,1,2} &= p_{0,1,2} + p_{1,1,1}
\end{aligned}$$

Its fairly straightforward to solve these equations; however, one can also observe that one solution is to have all states have equal probability. That is, the solution is

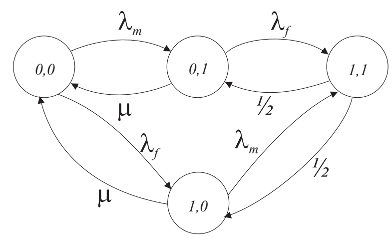
$$p_{0,0,0} = 1/5 \quad p_{1,0,1} = 1/5 \quad p_{0,1,2} = 1/5 \quad p_{1,1,1} = 1/5 \quad p_{1,1,2} = 1/5$$

(c) 15 Find the stationary probabilities p_{fms} for arbitrary (but nonzero) λ_f, λ_m, μ_f, and μ_m. What is the average number $E[M]$ of mechanics in the system at an arbitrary time?
In the case of switching, we found the general form of the stationary probabilities in the previous part. For the non-switching case, the general form of the stationary probabilities is a giant mess. Anyone who wrote the global balance equations, received a score proportional to their effort on this part. The number of mechanics in the system is 1 in states $(0, 1, 2)$, *or* $(1, 1, 1)$, *or* $(1, 1, 2)$ *and zero in all other states. The average number of mechanics in the system is*

$$E\left[M'\right] = p_{0,1,2} + p_{1,1,1} + p_{1,1,2}$$

(d) *15 points* Suppose instead, that when both a farmer and a mechanic are in the bank, each is served at the same time at rate 1/2 customers/minute. Draw the Markov chain for this system. (You can omit the variable s from your state.) For for arbitrary (but nonzero) λ_f, λ_m, μ_f, and μ_m, what are the stationary probabilities and what is the average number of mechanics $E[M']$ in this system?

In this case, when one of each kind of customer is in the queue, we don't need to keep track of who is in service. However, this part only makes complete sense if either $\mu_f = \mu_m = 1$ and the server devotes half its effort to each customer when two customers are in the system or if $\mu_m = \mu_f = 1/2$ and the server works twice as fast when there are two customers in the queue. In both cases, $\mu_f = \mu_m = \mu$ and the Markov chain is



Letting $\rho_m = \lambda_m/\mu$ and $\rho_f = \lambda_f/\mu$, solving for the stationary probabilities yields

$$p_{0,1} = (\lambda_m/\mu)p_{0,0} \quad p_{1,0} = (\lambda_f/\mu)p_{0,0} \quad p_{1,1} = (2\lambda_m\lambda_f/\mu)p_{0,0}$$

Choosing $p_{0,0}$ so the state probabilities add up to one yields

$$p_{0,0} = \frac{\mu}{\mu + \lambda_m + \lambda_f + 2\lambda_f\lambda_m}$$

The number of mechanics in the system is 1 in states $(0,1)$ or $(1,1)$ and zero in all other states. The average number of mechanics in the system is

$$E[M'] = p_{0,1} + p_{1,1} = \frac{\lambda_f + 2\lambda_f\lambda_m}{\mu + \lambda_m + \lambda_f + 2\lambda_f\lambda_m}$$

2. *50 points* This problem may seem familiar ... Trades of a particular stock occur as a Poisson process at a rate of $\lambda = 12$ trades/minute. On each trade, the stock price is equally likely to either go up 1/16 (an uptick), stay the same (a zero tick), or go down 1/16 (a downtick), independent of the number of trades and the outcome of any other trade. After t hours of trading, let $N(t)$ denote the number of trades made and let $X(t)$ equal the change in the stock price.

(a) *15 points* Let $U(t)$, $Z(t)$ and $D(t)$ denote the number of upticks, even ticks, and downticks after t hours. What is the joint PMF of $U(1)$, $Z(2)$, and $D(3)$?

This problem is the three way equivalent of a Bernoulli decomposition of the Poisson tick arrival process. For example, we can imagine performing the experiment by first deciding with probability 1/3 that we have a zero tick or with probability 2/3 that we have a nonzero tick. Then for each nonzero tick, we decide with equal probabilities to call it either an uptick or a downtick. The zero ticks and nonzero ticks would be independent Poisson arrival processes with rates 4 and 8 trades/minute. Further, since we perform Bernoulli decomposition on the nonzero tick process, the uptick and downtick processes are independent Poisson processes, each of rate 4 customers per minute. Since t hours is $60t$ minutes, $U(1)$, $Z(2)$ and $D(3)$ are independent Poisson random variables with mean values $E[U(1)] = 240$, $E[Z(2)] = 480$, and $E[D(3)] = 720$ trades. The joint PMF is

$$P_{U(1),Z(2),D(3)}(u,z,d) = \begin{cases} \frac{240^u e^{-240} 480^z e^{-480} 720^d e^{-720}}{u!z!d!} & u = 0,1,\ldots; z = 0,1,\ldots; d = 0,1,\ldots \\ 0 & \textit{otherwise} \end{cases}$$

(b) *15 points* A computerized trading program has developed a technique for earning a profit of \$0.01 on each uptick or downtick but a loss of \$0.01 on each zero tick. What is the expected revenue $E[R]$ per trade? Let $R(t)$ equal the profit of the trading program after t hours. What is the time average hourly reward rate $\lim_{t\to\infty} R(t)/t$?

On an individual trade, the revenue is $R = 0.01$ for an uptick or a downtick or $R = -0.01$ for a zero tick. Since up, down, and zero ticks are equally likely, the expected revenue on one trade is

$$E[R] = (2/3)(0.01) + (1/3)(-0.01) = 0.01/3$$

since trades occur as a Poisson process, a renewal occurs after each trade. During the ith renewal period, we earn a reward R_i for the ith trade. Since renewals occur at an average rate of 12 trades/minute or 720 trades/hour, the expected time between trades is $E[X] = 1/720$ hours. By the renewal reward theorem, the average hourly reward rate is

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]} = \frac{0.01/3}{1/720} = 2.40 \quad \$/hr$$

Since this about \$19.20 per day, probably the computerized trading program needs more work!

(c) *10 points* Can you find either the PMF or the MGF of $R(8)$?

We observe that $R(8) = 0.01[U(8) + D(8) - Z(8)]$. Since $U(8)$, $D(8)$, and $Z(8)$ are iid Poisson random variables with expected value of $4 \cdot 60 \cdot 8 = 1920$ trades,

$$\phi_{U(8)}(s) = \phi_{D(8)}(s) = \phi_{Z(8)}(s) = e^{1920(e^s-1)}$$

Next we observe that since $U(8)$, $D(8)$, and $Z(8)$ are independent,

$$\begin{aligned}
\phi_{R(8)}(s) &= E\left[e^{sR(8)}\right] = E\left[e^{(0.01s)[U(8)+D(8)-Z(8)]}\right] \\
&= E\left[e^{(0.01s)U(8)}\right] E\left[e^{(0.01s)D(8)}\right] E\left[e^{(-0.01s)Z(8)}\right] \\
&= \phi_{U(8)}(0.01s)\phi_{D(8)}(0.01s)\phi_{Z(8)}(-0.01s) \\
&= e^{1920(2e^{0.01s}+e^{-0.01s}-3)}
\end{aligned}$$

You can find the PMF if you like.

(d) *10 points* What are the mean and variance of $R(8)$?

You can solve either as a random sum of random variables where each trades has a random reward R, or by using the formula $R(8) = 0.01(U(8) + D(8) - Z(8))$. In the latter case, we have

$$\begin{aligned}
E[R(8)] &= 0.01\left(E[U(8)] + E[D(8)] - E[Z(8)]\right) = 19.20 \\
\sigma^2_{R(8)} &= 10^{-4}\left(\sigma^2_{U(8)} + \sigma^2_{D(8)} + \sigma^2_{Z(8)}\right) = 0.576
\end{aligned}$$

3. *50 points* For certain constants a, b, c, and d, random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = de^{-(a^2x^2+bxy+c^2y^2)}$$

(a) *20 points* Assuming a, b, c, and d are chosen correctly to guarantee a valid joint PDF, what are $E[X]$, $E[Y]$, σ^2_X and σ^2_Y?

The given joint PDF is an example of a bivariate Gaussian PDF if the constants a, b, c, and d are chosen correctly. Directly from Definition 5.10 of of the bivariate Gaussian PDF, we note that $E[X] = E[Y] = 0$. *In addition, we must have*

$$a^2 = \frac{1}{2\sigma_X^2(1-\rho^2)} \qquad c^2 = \frac{1}{2\sigma_Y^2(1-\rho^2)}$$
$$b = \frac{-\rho}{\sigma_X\sigma_Y(1-\rho^2)} \qquad d = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

We can solve for σ_X *and* σ_Y, *yielding*

$$\sigma_X = \frac{1}{a\sqrt{2(1-\rho^2)}} \qquad \sigma_Y = \frac{1}{c\sqrt{2(1-\rho^2)}}$$

Thus,

$$b = \frac{-\rho}{\sigma_X\sigma_Y(1-\rho^2)} = -2ac\rho$$

Hence,

$$\rho = \frac{-b}{2ac}$$

This implies

$$\sigma_X^2 = \frac{1}{2a^2(1-\rho^2)} = \frac{2c^2}{4a^2c^2-b^2} \qquad \sigma_Y^2 = \frac{1}{2c^2(1-\rho^2)} = \frac{2a^2}{4a^2c^2-b^2}$$

(b) *30 points* Under what conditions on the constants a, b, c, and d, is $f_{X,Y}(x,y)$ a valid joint PDF?

Since the given PDF is written in terms of a^2 *and* b^2, *a and b can be chosen arbitrarily. However, d must be nonnegative, and b must be chosen so that the correlation coefficient has magnitude less than 1. Thus,* $\rho^2 < 1$ *implies* $b^2 < 4a^2c^2$. *From part (a), we know that this conditions guarantees that the variances* σ_X^2 *and* σ_Y^2 *exist. Lastly, we observe that*

$$d^2 = \frac{1}{4\pi^2\sigma_X^2\sigma_Y^2(1-\rho^2)} = (1-\rho^2)a^2c^2 = a^2c^2 - b^2/4$$

Thus choosing $b^2 < 4a^2c^2$ *guarantees a valid PDF.*

4. *60 points* $X_1, X_2, \ldots$ is a sequence of iid $N[0,1]$ random variables. $Y_1, Y_2, \ldots$ is a random sequence such that $Y_n = (X_1 + \cdots + X_n)/n$.

(a) *10 points* Find either the PDF or MGF of Y_n.

Since $X_1, X_2, \ldots$ *are iid Gaussian,* Y_n *is a Gaussian. Since* Y_n *is the sample mean of* $X_1, \ldots, X_n$, $E[Y_n] = 0$ *and* $\sigma_{Y_n}^2 = \sigma_{X_n}^2/n = 1/n$.

(b) *20 points* Use the Chebyshev bound to upper bound $P\{Y_n > y\}$. Next use the Chernoff bound to upper bound $P\{Y_n > y\}$.

We will assume that $y \geq 0$. *Otherwise we cannot do better than the trivial bound* $P\{Y_n > y\} \leq 1$. *For* $y \geq 0$, *we observe that the Chebyshev bound is*

$$P\{Y_n > y\} = \frac{P\{|Y_n| > y\}}{2} \leq \frac{\sigma_{Y_n}^2}{2y^2} = \frac{1}{2ny^2}$$

Since Y_n *is Gaussian and* $\sigma_{Y_n}^2 = 1/n$, $\phi_{Y_n}(s) = e^{s^2/2n}$. *The Chernoff bound says that*

$$P\{Y_n > y\} \leq \min_{s\geq 0} e^{-sy}\phi_{Y_n}(s) = \min_{s\geq 0} e^{s^2/2n - sy}$$

To minimize the expression, it is sufficient to minimize the exponent. For nonnegative y, the minimizing s is $s = ny$. In this case,

$$P\{Y_n > y\} \leq e^{-ny^2/2}$$

For negative y, we cannot obtain a better bound than $P\{Y_n > y\} \leq 1$.

(c) *10 points* Find the joint PDF $f_{Y_n,Y_{n+1}}(y_1, y_2)$ of Y_n and Y_{n+1}.

Since Y_n and Y_{n+1} are derived from the same set of iid Gaussian random variables, Y_n and Y_{n+1} are bivariate Gaussian random variables. To find the joint PDF, we need to find the moments and the correlation coefficient. We already know that

$$E[Y_n] = E[Y_{n+1}] = 0 \qquad \sigma^2_{Y_n} = \frac{1}{n} \qquad \sigma^2_{Y_{n+1}} = \frac{1}{n+1}$$

Next we observe that

$$Y_{n+1} = \frac{n}{n+1} Y_n + \frac{1}{n+1} X_{n+1}$$

Since Y_n is a sum over $X_1, \ldots, X_n$, Y_n and X_{n+1} are independent and $E[Y_n X_{n+1}] = 0$. This implies

$$Cov[Y_n, Y_{n+1}] = E[Y_n Y_{n+1}] = \frac{n}{n+1} E\left[Y_n^2\right] + \frac{E[Y_n X_{n+1}]}{n+1} = \frac{1}{n+1}$$

Finally, Y_n and Y_{n+1} have correlation coefficient,

$$\rho = \frac{Cov[Y_n, Y_{n+1}]}{\sqrt{\sigma^2_{Y_n} \sigma^2_{Y_{n+1}}}} = \sqrt{\frac{n}{n+1}}$$

Plugging these values into Definition 5.10 yields the final result

$$f_{Y_n,Y_{n+1}}(y, y_1) = \frac{(n+1)\sqrt{n}}{2\pi} e^{-(n+1)[ny^2 - 2nyy_1 + (n+1)y_1]/2}$$

(d) *20 points* Suppose we use Y_n to predict Y_{n+1}. Let $Y = Y_n$ and let $X = Y_{n+1}$. Find the minimum mean square error (MMSE) estimator $\hat{X}_M(Y)$ and the optimal linear estimator (LMSE) estimator $\hat{X}_L(Y)$ of X given Y.

Since Y_n and Y_{n+1} are jointly Gaussian, we know that the LMSE and MMSE estimators are the same. Since both $E[X] = E[Y] = 0$, Theorem 9.11 tells us that

$$\hat{X}_M(Y) = \hat{X}_L(Y) = \rho \frac{\sigma_X}{\sigma_Y} Y$$

Note that

$$\sigma_X = 1/\sqrt{n+1} \qquad \sigma_Y = 1/\sqrt{n} \qquad \rho = \sqrt{\frac{n}{n+1}}$$

Thus

$$\hat{X}_M(Y) = \hat{X}_L(Y) = \frac{n}{n+1} Y$$

A second (and perhaps simpler) way to get the same result is to recall that

$$Y_{n+1} = \frac{n}{n+1} Y_n + \frac{1}{n+1} X_{n+1}$$

The MMSE estimator is also the conditional mean of Y_{n+1} given Y_n. In this case, the conditional mean is

$$E[Y_{n+1}|Y_n] = \frac{n}{n+1} E[Y_n|Y_n] + \frac{1}{n+1} E[X_{n+1}|Y_n] = \frac{n}{n+1} Y_n$$

where once again we used the independence of X_{n+1} and Y_n.