330:541 **Stochastic Signals and Systems** **Final Exam**

December 21, 1998

You have 180 minutes to answer the following problems. The point value of each part is indicated. You may use two sides of two sheets of notes. Please make sure that you have included your name, ID number and signature in each book used. Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. *60 points* Farmers and Mechanics Bank is a very peculiar bank that serves both kinds of customers: farmers and mechanics. Farmers and mechanics arrive as independent Poisson processes of rate λ_f and λ_m customers/minute. However, if there already is one farmer in the bank, all other farmers are blocked from entering. Similarly, no more than one mechanic can enter the bank at a time. Each blocked customer immediately departs, never to be seen again. Each farmer or mechanic that enters the banks has an independent exponential service requirement with mean $1/\mu_f$ minutes for a farmer or $1/\mu_m$ minutes for a mechanic. The bank has only a single teller. If there is one customer in the bank, that customer is served. With both a farmer and a mechanic in the bank, the teller swiches back and forth between the two customers, spending an independent exponential time with an average of 1 minute (assuming the customer doesn't finish service and leave) with each customer before switching.

   (a) *15 points* Let the 3-tuple fms denote the state of the system where f is the number of farmers and m is the number of mechanics in the bank, and s is the type of customer currently in service served. That is, $s = 0$ if the system is empty, $s = 1$ if a farmer is being served and $s = 2$ is a mechanic is being served. Draw a continuous time Markov chain for the system. How many communicating classes does this chain have?

   (b) *15 points* Find the stationary probabilities p_{fms} when $\lambda_f = 1 = \lambda_m = \mu_f = \mu_m = 1$. Hint: perhaps for neighboring states, fms and $f'm's'$, $p_{fms}p_{fms,f'm's'} = p_{f'm's'}p_{f'm's',fms}$?

   (c) 15 Find the stationary probabilities p_{fms} for arbitrary (but nonzero) λ_f, λ_m, μ_f, and μ_m. What is the average number $E[M]$ of mechanics in the system at an arbitrary time?

   (d) *15 points* Suppose instead, that when both a farmer and a mechanic are in the bank, each is served at the same time at rate $1/2$ customers/minute. Draw the Markov chain for this system. (You can omit the variable s from your state.) For for arbitrary (but nonzero) λ_f, λ_m, μ_f, and μ_m, what are the stationary probabilities and what is the average number of mechanics $E[M]$ in this system?

2. *50 points* This problem may seem familiar ... Trades of a particular stock occur as a Poisson process at a rate of $\lambda = 12$ trades/minute. On each trade, the stock price is equally likely to either go up $1/16$ (an uptick), stay the same (a zero tick), or go down $1/16$ (a downtick), independent of the number of trades and the outcome of any other trade. After t hours of trading, let $N(t)$ denote the number of trades made and let $X(t)$ equal the change in the stock price.

   (a) *15 points* Let $U(t)$, $Z(t)$ and $D(t)$ denote the number of upticks, even ticks, and downticks after t hours. What is the joint PMF of $U(1)$, $Z(2)$, and $D(3)$?

   (b) *15 points* A computerized trading program has developed a technique for earning a profit of \$0.01 on each uptick or downtick but a loss of \$0.01 on each zero tick. What is the expected revenue $E[R]$ per trade? Let $R(t)$ equal the profit of the trading program after t hours. What is the time average hourly reward rate $\lim_{t\to\infty} R(t)/t$?

   (c) *10 points* Can you find either the PMF or the MGF of $R(8)$?

   (d) *10 points* What are the mean and variance of $R(8)$?

3. *50 points* For certain constants a, b, c, and d, random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = de^{-(a^2x^2+bxy+c^2y^2)}$$

(a) *20 points* Assuming a, b, c, and d are chosen correctly to guarantee a valid joint PDF, what are $E[X]$, $E[Y]$, σ_X^2 and σ_Y^2?

(b) *30 points* Under what conditions on the constants a, b, c, and d, is $f_{X,Y}(x,y)$ a valid joint PDF?

4. *60 points* $X_1, X_2, \ldots$ is a sequence of iid $N[0,1]$ random variables. $Y_1, Y_2, \ldots$ is a random sequence such that $Y_n = (X_1 + \cdots + X_n)/n$.

(a) *10 points* Find either the PDF or MGF of Y_n.

(b) *20 points* Use the Chebyshev bound to upper bound $P\{Y_n > y\}$. Next use the Chernoff bound to upper bound $P\{Y_n > y\}$.

(c) *10 points* Find the joint PDF $f_{Y_n,Y_{n+1}}(y_1,y_2)$ of Y_n and Y_{n+1}.

(d) *20 points* Suppose we use Y_n to predict Y_{n+1}. Let $Y = Y_n$ and let $X = Y_{n+1}$. Find the minimum mean square error (MMSE) estimator $\hat{X}_M(Y)$ and the optimal linear estimator (LMSE) estimator $\hat{X}_L(Y)$ of X given Y.