Stochastic Signals and Systems
Problem 8.4.4 Solution
(Omitted from the PS8 Solution)

Problem Solution : Yates and Goodman, 8.4.4

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Since the a priori probabilities $P[H_0]$ and $P[H_1]$ are unknown, we use a Neyamn-Pearson formulation to find the ROC. For a threshold γ, the decision rule is

$$x \in A_0 \text{ if } \frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} \geq \gamma; \qquad x \in A_1 \text{ otherwise.} \tag{1}$$

Using the given conditional PDFs, we obtain

$$x \in A_0 \text{ if } e^{-(8x-x^2)/16} \geq \gamma x/4; \qquad x \in A_1 \text{ otherwise.} \tag{2}$$

Taking logarithms yields

$$x \in A_0 \text{ if } x^2 - 8x \geq 16\ln(\gamma/4) + 16\ln x; \qquad x \in A_1 \text{ otherwise.} \tag{3}$$

With some more rearranging,

$$x \in A_0 \text{ if } (x-4)^2 \geq \underbrace{16\ln(\gamma/4) + 16}_{\gamma_0} + 16\ln x; \qquad x \in A_1 \text{ otherwise.} \tag{4}$$

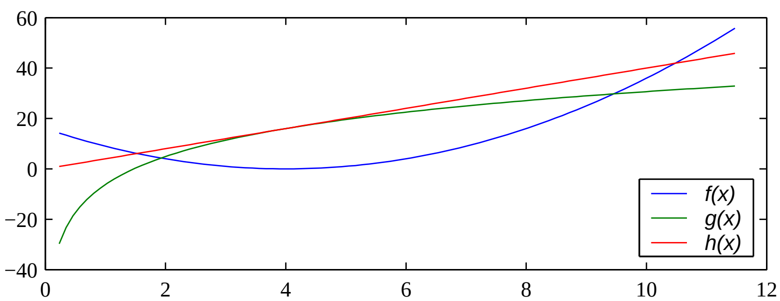
When we plot the functions $f(x) = (x-4)^2$ and $g(x) = \gamma_0 + 16\ln x$, we see that there exist x_1 and x_2 such that $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$. In terms of x_1 and x_2,

$$A_0 = [0, x_1] \cup [x_2, \infty), \qquad A_1 = (x_1, x_2). \tag{5}$$

Using a Taylor series expansion of $\ln x$ around $x = x_0 = 4$, we can show that

$$g(x) = \gamma_0 + 16\ln x \leq h(x) = \gamma_0 + 16(\ln 4 - 1) + 4x. \tag{6}$$

Since $h(x)$ is linear, we can use the quadratic formula to solve $f(x) = h(x)$, yielding a solution $\bar{x}_2 = 6 + \sqrt{4 + 16\ln 4 + \gamma_0}$. One can show that $x_2 \leq \bar{x}_2$. In the example shown below corresponding to $\gamma = 1$ shown here, $x_1 = 1.95$, $x_2 = 9.5$ and $\bar{x}_2 = 6 + \sqrt{20} = 10.47$.



Given x_1 and x_2, the false alarm and miss probabilities are

$$P_{\rm FA} = P[A_1|H_0] = \int_{x_1}^{2} \frac{1}{2} e^{-x/2}\,dx = e^{-x_1/2} - e^{-x_2/2}, \tag{7}$$

$$P_{\rm MISS} = 1 - P[A_1|H_1] = 1 - \int_{x_1}^{x_2} \frac{x}{8} e^{-x^2/16}\,dx = 1 - e^{-x_1^2/16} + e^{-x_2^2/16} \tag{8}$$

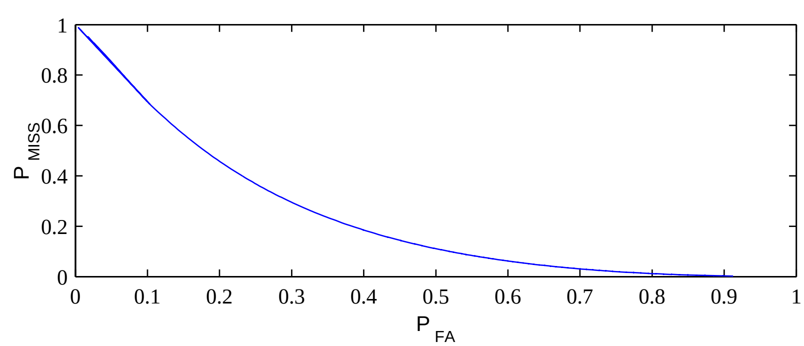
To calculate the ROC, we need to find x_1 and x_2. Rather than find them exactly, we calculate $f(x)$ and $g(x)$ for discrete steps over the interval $[0, 1+\bar{x}_2]$ and find the discrete values closest to x_1 and x_2. However, for these approximations to x_1 and x_2, we calculate the exact false alarm and miss probabilities. As a result, the optimal detector using the exact x_1 and x_2 cannot be worse than the ROC that we calculate.

In terms of MATLAB, we divide the work into the functions `gasroc(n)` which generates the ROC by calling `[x1,x2]=gasrange(gamma)` to calculate x_1 and x_2 for a given value of γ.

```
function [pfa,pmiss]=gasroc(n);
a=(400)^(1/(n-1));
k=1:n;
g=0.05*(a.^(k-1));
pfa=zeros(n,1);
pmiss=zeros(n,1);
for k=1:n,
  [x1,x2]=gasrange(g(k));
  pmiss(k)=1-(exp(-x1^2/16)...
    -exp(-x2^2/16));
  pfa(k)=exp(-x1/2)-exp(-x2/2);
end
plot(pfa,pmiss);
ylabel('P_{\rm MISS}');
xlabel('P_{\rm FA}');
```

```
function [x1,x2]=gasrange(gamma);
g=16+16*log(gamma/4);
xmax=7+ ...
  sqrt(max(0,4+(16*log(4))+g));
dx=xmax/500;
x=dx:dx:4;
y=(x-4).^2-g-16*log(x);
[ym,i]=min(abs(y));
x1=x(i);
x=4:dx:xmax;
y=(x-4).^2-g-16*log(x);
[ym,i]=min(abs(y));
x2=x(i);
```

The argment `n` of `gasroc(n)` generates the ROC for `n` values of γ, ranging from from 1/20 to 20 in multiplicative steps. Here is the resulting ROC:



After all of this work, we see that the sensor is not particularly good in the the ense that no matter how we choose the thresholds, we cannot reduce both the miss and false alarm probabilities under 30 percent.