ECE541
Stochastic Signals and Systems
Problem Set 7

Problem Solutions : Yates and Goodman, 7.1.3 7.2.2 7.2.4 7.3.3 7.3.4 7.3.6 7.4.2 and 7.4.6

Problem 7.1.3 Solution

This problem is in the wrong section since the *standard error* isn't defined until Section 7.3. However is we peek ahead to this section, the problem isn't very hard. Given the sample mean estimate $M_n(X)$, the standard error is defined as the standard deviation $e_n = \sqrt{\text{Var}[M_n(X)]}$. In our problem, we use samples X_i to generate $Y_i = X_i^2$. For the sample mean $M_n(Y)$, we need to find the standard error

$$e_n = \sqrt{\text{Var}[M_n(Y)]} = \sqrt{\frac{\text{Var}[Y]}{n}}. \tag{1}$$

Since X is a uniform $(0, 1)$ random variable,

$$E[Y] = E\left[X^2\right] = \int_0^1 x^2\,dx = 1/3, \tag{2}$$

$$E\left[Y^2\right] = E\left[X^4\right] = \int_0^1 x^4\,dx = 1/5. \tag{3}$$

Thus $\text{Var}[Y] = 1/5 - (1/3)^2 = 4/45$ and the sample mean $M_n(Y)$ has standard error

$$e_n = \sqrt{\frac{4}{45n}}. \tag{4}$$

Problem 7.2.2 Solution

We know from the Chebyshev inequality that

$$P[|X - E[X]| \geq c] \leq \frac{\sigma_X^2}{c^2} \tag{1}$$

Choosing $c = k\sigma_X$, we obtain

$$P[|X - E[X]| \geq k\sigma] \leq \frac{1}{k^2} \tag{2}$$

The actual probability the Gaussian random variable Y is more than k standard deviations from its expected value is

$$P[|Y - E[Y]| \geq k\sigma_Y] = P[Y - E[Y] \leq -k\sigma_Y] + P[Y - E[Y] \geq k\sigma_Y] \tag{3}$$

$$= 2P\left[\frac{Y - E[Y]}{\sigma_Y} \geq k\right] \tag{4}$$

$$= 2Q(k) \tag{5}$$

The following table compares the upper bound and the true probability:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Chebyshev bound	1	0.250	0.111	0.0625	0.040
$2Q(k)$	0.317	0.046	0.0027	6.33×10^{-5}	5.73×10^{-7}

(6)

The Chebyshev bound gets increasingly weak as k goes up. As an example, for $k = 4$, the bound exceeds the true probability by a factor of 1,000 while for $k = 5$ the bound exceeds the actual probability by a factor of nearly 100,000.

Problem 7.2.4 Solution

On each roll of the dice, a success, namely snake eyes, occurs with probability $p = 1/36$. The number of trials, R, needed for three successes is a Pascal $(k = 3, p)$ random variable with

$$E[R] = 3/p = 108, \qquad \text{Var}[R] = 3(1-p)/p^2 = 3780. \tag{1}$$

(a) By the Markov inequality,

$$P[R \geq 250] \leq \frac{E[R]}{250} = \frac{54}{125} = 0.432. \tag{2}$$

(b) By the Chebyshev inequality,

$$P[R \geq 250] = P[R - 108 \geq 142] = P[|R - 108| \geq 142] \tag{3}$$

$$\leq \frac{\text{Var}[R]}{(142)^2} = 0.1875. \tag{4}$$

(c) The exact value is $P[R \geq 250] = 1 - \sum_{r=3}^{249} P_R(r)$. Since there is no way around summing the Pascal PMF to find the CDF, this is what `pascalcdf` does.

```
>> 1-pascalcdf(3,1/36,249)
ans =
    0.0299
```

Thus the Markov and Chebyshev inequalities are valid bounds but not good estimates of $P[R \geq 250]$.

Problem 7.3.3 Solution

This problem is really very simple. If we let $Y = X_1X_2$ and for the ith trial, let $Y_i = X_1(i)X_2(i)$, then $\hat{R}_n = M_n(Y)$, the sample mean of random variable Y. By Theorem 7.5, $M_n(Y)$ is unbiased. Since $\text{Var}[Y] = \text{Var}[X_1X_2] < \infty$, Theorem 7.7 tells us that $M_n(Y)$ is a consistent sequence.

Problem 7.3.4 Solution

(a) Since the expectation of a sum equals the sum of the expectations also holds for vectors,

$$E[\mathbf{M}(n)] = \frac{1}{n}\sum_{i=1}^{n} E[\mathbf{X}(i)] = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\mu}_{\mathbf{X}} = \boldsymbol{\mu}_{\mathbf{X}}. \tag{1}$$

(b) The jth component of $\mathbf{M}(n)$ is $M_j(n) = \frac{1}{n}\sum_{i=1}^{n} X_j(i)$, which is just the sample mean of X_j. Defining $A_j = \{|M_j(n) - \mu_j| \geq c\}$, we observe that

$$P\left[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c\right] = P[A_1 \cup A_2 \cup \cdots \cup A_k]. \tag{2}$$

Applying the Chebyshev inequality to $M_j(n)$, we find that

$$P[A_j] \leq \frac{\operatorname{Var}[M_j(n)]}{c^2} = \frac{\sigma_j^2}{nc^2}. \tag{3}$$

By the union bound,

$$P\left[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c\right] \leq \sum_{j=1}^{k} P[A_j] \leq \frac{1}{nc^2}\sum_{j=1}^{k} \sigma_j^2 \tag{4}$$

Since $\sum_{j=1}^{k} \sigma_j^2 < \infty$, $\lim_{n\to\infty} P[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c] = 0$.

Problem 7.3.6 Solution

(a) From Theorem 6.2, we have

$$\operatorname{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \operatorname{Var}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \operatorname{Cov}[X_i, X_j] \tag{1}$$

Note that $\operatorname{Var}[X_i] = \sigma^2$ and for $j > i$, $\operatorname{Cov}[X_i, X_j] = \sigma^2 a^{j-i}$. This implies

$$\operatorname{Var}[X_1 + \cdots + X_n] = n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a^{j-i} \tag{2}$$

$$= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1}\left(a + a^2 + \cdots + a^{n-i}\right) \tag{3}$$

$$= n\sigma^2 + \frac{2a\sigma^2}{1-a}\sum_{i=1}^{n-1}(1 - a^{n-i}) \tag{4}$$

With some more algebra, we obtain

$$\mathrm{Var}[X_1 + \cdots + X_n] = n\sigma^2 + \frac{2a\sigma^2}{1-a}(n-1) - \frac{2a\sigma^2}{1-a}\left(a + a^2 + \cdots + a^{n-1}\right) \quad (5)$$

$$= \left(\frac{n(1+a)\sigma^2}{1-a}\right) - \frac{2a\sigma^2}{1-a} - 2\sigma^2\left(\frac{a}{1-a}\right)^2 (1 - a^{n-1}) \quad (6)$$

Since $a/(1-a)$ and $1 - a^{n-1}$ are both nonnegative,

$$\mathrm{Var}[X_1 + \cdots + X_n] \le n\sigma^2 \left(\frac{1+a}{1-a}\right) \quad (7)$$

(b) Since the expected value of a sum equals the sum of the expected values,

$$E\left[M(X_1, \ldots, X_n)\right] = \frac{E\left[X_1\right] + \cdots + E\left[X_n\right]}{n} = \mu \quad (8)$$

The variance of $M(X_1, \ldots, X_n)$ is

$$\mathrm{Var}[M(X_1, \ldots, X_n)] = \frac{\mathrm{Var}[X_1 + \cdots + X_n]}{n^2} \le \frac{\sigma^2(1+a)}{n(1-a)} \quad (9)$$

Applying the Chebyshev inequality to $M(X_1, \ldots, X_n)$ yields

$$P\left[|M(X_1, \ldots, X_n) - \mu| \ge c\right] \le \frac{\mathrm{Var}[M(X_1, \ldots, X_n)]}{c^2} \le \frac{\sigma^2(1+a)}{n(1-a)c^2} \quad (10)$$

(c) Taking the limit as n approaches infinity of the bound derived in part (b) yields

$$\lim_{n\to\infty} P\left[|M(X_1, \ldots, X_n) - \mu| \ge c\right] \le \lim_{n\to\infty} \frac{\sigma^2(1+a)}{n(1-a)c^2} = 0 \quad (11)$$

Thus

$$\lim_{n\to\infty} P\left[|M(X_1, \ldots, X_n) - \mu| \ge c\right] = 0 \quad (12)$$

Problem 7.4.2 Solution

$X_1, X_2, \ldots$ are iid random variables each with mean 75 and standard deviation 15.

(a) We would like to find the value of n such that

$$P\left[74 \le M_n(X) \le 76\right] = 0.99 \quad (1)$$

When we know only the mean and variance of X_i, our only real tool is the Chebyshev inequality which says that

$$P\left[74 \le M_n(X) \le 76\right] = 1 - P\left[|M_n(X) - E\left[X\right]| \ge 1\right] \quad (2)$$

$$\ge 1 - \frac{\mathrm{Var}\left[X\right]}{n} = 1 - \frac{225}{n} \ge 0.99 \quad (3)$$

This yields $n \ge 22{,}500$.

(b) If each X_i is a Gaussian, the sample mean, $M_n(X)$ will also be Gaussian with mean and variance

$$E[M_{n'}(X)] = E[X] = 75 \tag{4}$$
$$\mathrm{Var}[M_{n'}(X)] = \mathrm{Var}[X]/n' = 225/n' \tag{5}$$

In this case,

$$P[74 \leq M_{n'}(X) \leq 76] = \Phi\left(\frac{76-\mu}{\sigma}\right) - \Phi\left(\frac{74-\mu}{\sigma}\right) \tag{6}$$
$$= \Phi(\sqrt{n'}/15) - \Phi(-\sqrt{n'}/15) \tag{7}$$
$$= 2\Phi(\sqrt{n'}/15) - 1 = 0.99 \tag{8}$$

Thus, $n' = 1{,}521$.

Since even under the Gaussian assumption, the number of samples n' is so large that even if the X_i are not Gaussian, the sample mean may be approximated by a Gaussian. Hence, about 1500 samples probably is about right. However, in the absence of any information about the PDF of X_i beyond the mean and variance, we cannot make any guarantees stronger than that given by the Chebyshev inequality.

Problem 7.4.6 Solution

Both questions can be answered using the following equation from Example 7.6:

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{P[A](1-P[A])}{nc^2} \tag{1}$$

The unusual part of this problem is that we are given the true value of $P[A]$. Since $P[A] = 0.01$, we can write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{0.0099}{nc^2} \tag{2}$$

(a) In this part, we meet the requirement by choosing $c = 0.001$ yielding

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq 0.001\right] \leq \frac{9900}{n} \tag{3}$$

Thus to have confidence level 0.01, we require that $9900/n \leq 0.01$. This requires $n \geq 990{,}000$.

(b) In this case, we meet the requirement by choosing $c = 10^{-3}P[A] = 10^{-5}$. This implies

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{P[A](1-P[A])}{nc^2} = \frac{0.0099}{n10^{-10}} = \frac{9.9 \times 10^7}{n} \tag{4}$$

The confidence level 0.01 is met if $9.9 \times 10^7/n = 0.01$ or $n = 9.9 \times 10^9$.