ECE 541
Stochastic Signals and Systems
Problem Set 4 Solution

Problem Solutions : Yates and Goodman, 5.7.8 5.8.2 5.8.4 10.2.4 10.3.4 10.4.2 10.5.5 10.5.6 10.5.8 10.6.3 and 10.6.4

Problem 5.7.8 Solution

As given in the problem statement, we define the m-dimensional vector $\mathbf{X}$, the n-dimensional vector $\mathbf{Y}$ and $\mathbf{W} = \begin{bmatrix}\mathbf{X}' \\ \mathbf{Y}'\end{bmatrix}'$. Note that $\mathbf{W}$ has expected value

$$\boldsymbol{\mu}_{\mathbf{W}} = E[\mathbf{W}] = E\left[\begin{bmatrix}\mathbf{X} \\ \mathbf{Y}\end{bmatrix}\right] = \begin{bmatrix}E[\mathbf{X}] \\ E[\mathbf{Y}]\end{bmatrix} = \begin{bmatrix}\boldsymbol{\mu}_{\mathbf{X}} \\ \boldsymbol{\mu}_{\mathbf{Y}}\end{bmatrix}. \tag{1}$$

The covariance matrix of $\mathbf{W}$ is

$$\begin{aligned}
\mathbf{C}_{\mathbf{W}} &= E\left[(\mathbf{W}-\boldsymbol{\mu}_{\mathbf{W}})(\mathbf{W}-\boldsymbol{\mu}_{\mathbf{W}})'\right] && (2)\\
&= E\left[\begin{bmatrix}\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{Y}-\boldsymbol{\mu}_{\mathbf{Y}}\end{bmatrix}\begin{bmatrix}(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})' & (\mathbf{Y}-\boldsymbol{\mu}_{\mathbf{Y}})'\end{bmatrix}\right] && (3)\\
&= \begin{bmatrix}E[(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})'] & E[(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})(\mathbf{Y}-\boldsymbol{\mu}_{\mathbf{Y}})'] \\ E[(\mathbf{Y}-\boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})'] & E[(\mathbf{Y}-\boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{Y}-\boldsymbol{\mu}_{\mathbf{Y}})']\end{bmatrix} && (4)\\
&= \begin{bmatrix}\mathbf{C}_{\mathbf{X}} & \mathbf{C}_{\mathbf{XY}} \\ \mathbf{C}_{\mathbf{YX}} & \mathbf{C}_{\mathbf{Y}}\end{bmatrix}. && (5)
\end{aligned}$$

The assumption that $\mathbf{X}$ and $\mathbf{Y}$ are independent implies that

$$\mathbf{C}_{\mathbf{XY}} = E\left[(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})(\mathbf{Y}'-\boldsymbol{\mu}'_{\mathbf{Y}})\right] = (E[(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})]\,E\left[(\mathbf{Y}'-\boldsymbol{\mu}'_{\mathbf{Y}})\right] = \mathbf{0}. \tag{6}$$

This also implies $\mathbf{C}_{\mathbf{YX}} = \mathbf{C}'_{\mathbf{XY}} = \mathbf{0}'$. Thus

$$\mathbf{C}_{\mathbf{W}} = \begin{bmatrix}\mathbf{C}_{\mathbf{X}} & \mathbf{0} \\ \mathbf{0}' & \mathbf{C}_{\mathbf{Y}}\end{bmatrix}. \tag{7}$$

Problem 5.8.2 Solution

(a) The covariance matrix $\mathbf{C}_X$ has $\text{Var}[X_i] = 25$ for each diagonal entry. For $i \neq j$, the i,jth entry of $\mathbf{C}_X$ is

$$[\mathbf{C}_X]_{ij} = \rho_{X_iX_j}\sqrt{\text{Var}[X_i]\,\text{Var}[X_j]} = (0.8)(25) = 20 \tag{1}$$

The covariance matrix of X is a 10×10 matrix of the form

$$\mathbf{C}_X = \begin{bmatrix}25 & 20 & \cdots & 20 \\ 20 & 25 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 20 \\ 20 & \cdots & 20 & 25\end{bmatrix}. \tag{2}$$

(b) We observe that

$$Y = \begin{bmatrix}1/10 & 1/10 & \cdots & 1/10\end{bmatrix} \mathbf{X} = \mathbf{A}\mathbf{X} \tag{3}$$

Since Y is the average of 10 iid random variables, $E[Y] = E[X_i] = 5$. Since Y is a scalar, the 1×1 covariance matrix $\mathbf{C}_Y = \text{Var}[Y]$. By Theorem 5.13, the variance of Y is

$$\text{Var}[Y] = \mathbf{C}_Y = \mathbf{A}\mathbf{C}_X\mathbf{A}' = 20.5 \tag{4}$$

Since Y is Gaussian,

$$P[Y \le 25] = P\left[\frac{Y-5}{\sqrt{20.5}} \le \frac{25-20.5}{\sqrt{20.5}}\right] = \Phi(0.9939) = 0.8399. \tag{5}$$

Problem 5.8.4 Solution

The covariance matrix $\mathbf{C}_X$ has $\text{Var}[X_i] = 25$ for each diagonal entry. For $i \neq j$, the i,jth entry of $\mathbf{C}_X$ is

$$[\mathbf{C}_X]_{ij} = \rho_{X_iX_j}\sqrt{\text{Var}[X_i]\,\text{Var}[X_j]} = (0.8)(25) = 20 \tag{1}$$

The covariance matrix of X is a 10×10 matrix of the form

$$\mathbf{C}_X = \begin{bmatrix} 25 & 20 & \cdots & 20 \\ 20 & 25 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 20 \\ 20 & \cdots & 20 & 25 \end{bmatrix}. \tag{2}$$

A program to estimate $P[W \le 25]$ uses `gaussvector` to generate m sample vector of race times $\mathbf{X}$. In the program `sailboats.m`, `X` is an $10 \times m$ matrix such that each column of `X` is a vector of race times. In addition `min(X)` is a row vector indicating the fastest time in each race.

```
function p=sailboats(w,m)
%Usage: p=sailboats(f,m)
%In Problem 5.8.4, W is the
%winning time in a 10 boat race.
%We use m trials to estimate
%P[W<=w]
CX=(5*eye(10))+(20*ones(10,10));
mu=35*ones(10,1);
X=gaussvector(mu,CX,m);
W=min(X);
p=sum(W<=w)/m;
```

```
>> sailboats(25,10000)
ans =
    0.0827
>> sailboats(25,100000)
ans =
    0.0801
>> sailboats(25,100000)
ans =
    0.0803
>> sailboats(25,100000)
ans =
    0.0798
```

We see from repeated experiments with $m = 100{,}000$ trials that $P[W \le 25] \approx 0.08$.

Problem 10.2.4 Solution

The statement is *false*. As a counterexample, consider the rectified cosine waveform $X(t) = R|\cos 2\pi f t|$ of Example 10.9. When $t = \pi/2$, then $\cos 2\pi f t = 0$ so that $X(\pi/2) = 0$. Hence $X(\pi/2)$ has PDF

$$f_{X(\pi/2)}(x) = \delta(x) \tag{1}$$

That is, $X(\pi/2)$ is a discrete random variable.

Problem 10.3.4 Solution

Since the problem states that the pulse is delayed, we will assume $T \geq 0$. This problem is difficult because the answer will depend on t. In particular, for $t < 0$, $X(t) = 0$ and $f_{X(t)}(x) = \delta(x)$. Things are more complicated when $t > 0$. For $x < 0$, $P[X(t) > x] = 1$. For $x \geq 1$, $P[X(t) > x] = 0$. Lastly, for $0 \leq x < 1$,

$$P[X(t) > x] = P\left[e^{-(t-T)}u(t-T) > x\right] \tag{1}$$
$$= P[t + \ln x < T \leq t] \tag{2}$$
$$= F_T(t) - F_T(t + \ln x) \tag{3}$$

Note that condition $T \leq t$ is needed to make sure that the pulse doesn't arrive after time t. The other condition $T > t + \ln x$ ensures that the pulse didn't arrrive too early and already decay too much. We can express these facts in terms of the CDF of $X(t)$.

$$F_{X(t)}(x) = 1 - P[X(t) > x] = \begin{cases} 0 & x < 0 \\ 1 + F_T(t + \ln x) - F_T(t) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{4}$$

We can take the derivative of the CDF to find the PDF. However, we need to keep in mind that the CDF has a jump discontinuity at $x = 0$. In particular, since $\ln 0 = -\infty$,

$$F_{X(t)}(0) = 1 + F_T(-\infty) - F_T(t) = 1 - F_T(t) \tag{5}$$

Hence, when we take a derivative, we will see an impulse at $x = 0$. The PDF of $X(t)$ is

$$f_{X(t)}(x) = \begin{cases} (1 - F_T(t))\delta(x) + f_T(t + \ln x)/x & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 10.4.2 Solution

Each W_n is the sum of two identical independent Gaussian random variables. Hence, each W_n must have the same PDF. That is, the W_n are identically distributed. However, since W_{n-1} and W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We can verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30 \tag{1}$$

Next, we observe that W_{n-1} and W_n have covariance

$$\text{Cov}[W_{n-1}, W_n] = E[W_{n-1}W_n] - E[W_n]E[W_{n-1}] \tag{2}$$
$$= \frac{1}{4}E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900 \tag{3}$$

We observe that for $n \neq m$, $E[X_n X_m] = E[X_n]E[X_m] = 900$ while

$$E\left[X_n^2\right] = \text{Var}[X_n] + (E[X_n])^2 = 916 \tag{4}$$

Thus,

$$\text{Cov}[W_{n-1}, W_n] = \frac{900 + 916 + 900 + 900}{4} - 900 = 4 \tag{5}$$

Since $\text{Cov}[W_{n-1}, W_n] \neq 0$, W_n and W_{n-1} must be dependent.

Problem 10.5.5 Solution

Note that it matters whether $t \geq 2$ minutes. If $t \leq 2$, then any customers that have arrived must still be in service. Since a Poisson number of arrivals occur during $(0, t]$,

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq t \leq 2) \tag{1}$$

For $t \geq 2$, the customers in service are precisely those customers that arrived in the interval $(t-2, t]$. The number of such customers has a Poisson PMF with mean $\lambda[t - (t-2)] = 2\lambda$. The resulting PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (2\lambda)^n e^{-2\lambda}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (t \geq 2) \tag{2}$$

Problem 10.5.6 Solution

The time T between queries are independent exponential random variables with PDF

$$f_T(t) = \begin{cases} (1/8)e^{-t/8} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the PDF, we can calculate for $t > 0$,

$$P[T \geq t] = \int_0^t f_T(t')\, dt' = e^{-t/8} \tag{2}$$

Using this formula, each question can be easily answered.

(a) $P[T \geq 4] = e^{-4/8} \approx 0.951$.

(b)

$$P[T \geq 13 | T \geq 5] = \frac{P[T \geq 13, T \geq 5]}{P[T \geq 5]} \tag{3}$$

$$= \frac{P[T \geq 13]}{P[T \geq 5]} = \frac{e^{-13/8}}{e^{-5/8}} = e^{-1} \approx 0.368 \tag{4}$$

(c) Although the time betwen queries are independent exponential random variables, $N(t)$ is not exactly a Poisson random process because the first query occurs at time $t = 0$. Recall that in a Poisson process, the first arrival occurs some time after $t = 0$. However $N(t) - 1$ is a Poisson process of rate 8. Hence, for $n = 0, 1, 2, \ldots$,

$$P[N(t) - 1 = n] = (t/8)^n e^{-t/8}/n! \tag{5}$$

Thus, for $n = 1, 2, \ldots$, the PMF of $N(t)$ is

$$P_{N(t)}(n) = P[N(t) - 1 = n - 1] = (t/8)^{n-1} e^{-t/8}/(n-1)! \tag{6}$$

The complete expression of the PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (t/8)^{n-1} e^{-t/8}/(n-1)! & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 10.5.8 Solution

(a) For $X_i = -\ln U_i$, we can write

$$P[X_i > x] = P[-\ln U_i > x] = P[\ln U_i \le -x] = P\left[U_i \le e^{-x}\right] \tag{1}$$

When $x < 0$, $e^{-x} > 1$ so that $P[U_i \le e^{-x}] = 1$. When $x \ge 0$, we have $0 < e^{-x} \le 1$, implying $P[U_i \le e^{-x}] = e^{-x}$. Combining these facts, we have

$$P[X_i > x] = \begin{cases} 1 & x < 0 \\ e^{-x} & x \ge 0 \end{cases} \tag{2}$$

This permits us to show that the CDF of X_i is

$$F_{X_i}(x) = 1 - P[X_i > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x > 0 \end{cases} \tag{3}$$

We see that X_i has an exponential CDF with mean 1.

(b) Note that $N = n$ iff

$$\prod_{i=1}^{n} U_i \ge e^{-t} > \prod_{i=1}^{n+1} U_i \tag{4}$$

By taking the logarithm of both inequalities, we see that $N = n$ iff

$$\sum_{i=1}^{n} \ln U_i \ge -t > \sum_{i=1}^{n+1} \ln U_i \tag{5}$$

Next, we multiply through by -1 and recall that $X_i = -\ln U_i$ is an exponential random variable. This yields $N = n$ iff

$$\sum_{i=1}^{n} X_i \le t < \sum_{i=1}^{n+1} X_i \tag{6}$$

Now we recall that a Poisson process $N(t)$ of rate 1 has independent exponential interarrival times $X_1, X_2, \ldots$. That is, the ith arrival occurs at time $\sum_{j=1}^{i} X_j$. Moreover, $N(t) = n$ iff the first n arrivals occur by time t but arrival $n+1$ occurs after time t. Since the random variable $N(t)$ has a Poisson distribution with mean t, we can write

$$P\left[\sum_{i=1}^{n} X_i \le t < \sum_{i=1}^{n+1} X_i\right] = P[N(t) = n] = \frac{t^n e^{-t}}{n!}. \tag{7}$$

Problem 10.6.3 Solution

We start with the case when $t \geq 2$. When each service time is equally likely to be either 1 minute or 2 minutes, we have the following situation. Let M_1 denote those customers that arrived in the interval $(t-1, 1]$. All M_1 of these customers will be in the bank at time t and M_1 is a Poisson random variable with mean λ.

Let M_2 denote the number of customers that arrived during $(t-2, t-1]$. Of course, M_2 is Poisson with expected value λ. We can view each of the M_2 customers as flipping a coin to determine whether to choose a 1 minute or a 2 minute service time. Only those customers that chooses a 2 minute service time will be in service at time t. Let M_2' denote those customers choosing a 2 minute service time. It should be clear that M_2' is a Poisson number of Bernoulli random variables. Theorem 10.6 verifies that using Bernoulli trials to decide whether the arrivals of a rate λ Poisson process should be counted yields a Poisson process of rate $p\lambda$. A consequence of this result is that a Poisson number of Bernoulli (success probability p) random variables has Poisson PMF with mean $p\lambda$. In this case, M_2' is Poisson with mean $\lambda/2$. Moreover, the number of customers in service at time t is $N(t) = M_1 + M_2'$. Since M_1 and M_2' are independent Poisson random variables, their sum $N(t)$ also has a Poisson PMF. This was verified in Theorem 6.9. Hence $N(t)$ is Poisson with mean $E[N(t)] = E[M_1] + E[M_2'] = 3\lambda/2$. The PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (3\lambda/2)^n e^{-3\lambda/2}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (t \geq 2) \tag{1}$$

Now we can consider the special cases arising when $t < 2$. When $0 \leq t < 1$, every arrival is still in service. Thus the number in service $N(t)$ equals the number of arrivals and has the PMF

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq t \leq 1) \tag{2}$$

When $1 \leq t < 2$, let M_1 denote the number of customers in the interval $(t-1, t]$. All M_1 customers arriving in that interval will be in service at time t. The M_2 customers arriving in the interval $(0, t-1]$ must each flip a coin to decide one a 1 minute or two minute service time. Only those customers choosing the two minute service time will be in service at time t. Since M_2 has a Poisson PMF with mean $\lambda(t-1)$, the number M_2' of those customers in the system at time t has a Poisson PMF with mean $\lambda(t-1)/2$. Finally, the number of customers in service at time t has a Poisson PMF with expected value $E[N(t)] = E[M_1] + E[M_2'] = \lambda + \lambda(t-1)/2$. Hence, the PMF of $N(t)$ becomes

$$P_{N(t)}(n) = \begin{cases} (\lambda(t+1)/2)^n e^{-\lambda(t+1)/2}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (1 \leq t \leq 2) \tag{3}$$

Problem 10.6.4 Solution

Since the arrival times $S_1, \ldots, S_n$ are ordered in time and since a Poisson process cannot have two simultaneous arrivals, the conditional PDF $f_{S_1,\ldots,S_n|N}(S_1, \ldots, S_n|n)$ is nonzero only if $s_1 < s_2 < \cdots < s_n < T$. In this case, consider an arbitrarily small Δ; in particular, $\Delta < \min_i(s_{i+1} - s_i)/2$ implies that the intervals $(s_i, s_i + \Delta]$ are non-overlapping. We now find the joint probability

$$P[s_1 < S_1 \leq s_1 + \Delta, \ldots, s_n < S_n \leq s_n + \Delta, N = n]$$

that each S_i is in the interval $(s_i, s_i + \Delta]$ and that $N = n$. This joint event implies that there were zero arrivals in each interval $(s_i + \Delta, s_{i+1}]$. That is, over the interval $[0, T]$, the Poisson process has exactly one arrival in each interval $(s_i, s_i + \Delta]$ and zero arrivals in the time period $T - \bigcup_{i=1}^{n}(s_i, s_i + \Delta]$. The collection of intervals in which there was no arrival had a total duration of $T - n\Delta$. Note that the probability of exactly one arrival in the interval $(s_i, s_i + \Delta]$ is $\lambda\Delta e^{-\lambda\delta}$ and the probability of zero arrivals in a period of duration $T - n\Delta$ is $e^{-\lambda(T_n - \Delta)}$. In addition, the event of one arrival in each interval $(s_i, s_i + \Delta)$ and zero events in the period of length $T - n\Delta$ are independent events because they consider non-overlapping periods of the Poisson process. Thus,

$$P\left[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta, N = n\right] = \left(\lambda\Delta e^{-\lambda\Delta}\right)^n e^{-\lambda(T - n\Delta)} \tag{1}$$

$$= (\lambda\Delta)^n e^{-\lambda T} \tag{2}$$

Since $P[N = n] = (\lambda T)^n e^{-\lambda T}/n!$, we see that

$$P\left[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta | N = n\right]$$

$$= \frac{P\left[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta, N = n\right]}{P\left[N = n\right]} \tag{3}$$

$$= \frac{(\lambda\Delta)^n e^{-\lambda T}}{(\lambda T)^n e^{-\lambda T}/n!} \tag{4}$$

$$= \frac{n!}{T^n}\Delta^n \tag{5}$$

Finally, for infinitesimal Δ, the conditional PDF of $S_1, \ldots, S_n$ given $N = n$ satisfies

$$f_{S_1,\ldots,S_n|N}\left(s_1, \ldots, s_n | n\right)\Delta^n = P\left[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta | N = n\right] \tag{6}$$

$$= \frac{n!}{T^n}\Delta^n \tag{7}$$

Since the conditional PDF is zero unless $s_1 < s_2 < \cdots < s_n \le T$, it follows that

$$f_{S_1,\ldots,S_n|N}\left(s_1, \ldots, s_n | n\right) = \begin{cases} n!/T^n & 0 \le s_1 < \cdots < s_n \le T, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

If it seems that the above argument had some "hand-waving," we now do the derivation of $P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta | N = n]$ in somewhat excruciating detail. (Feel free to skip the following if you were satisfied with the earlier explanation.)

For the interval $(s, t]$, we use the shorthand notation $0_{(s,t)}$ and $1_{(s,t)}$ to denote the events of 0 arrivals and 1 arrival respectively. This notation permits us to write

$$P\left[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta, N = n\right]$$

$$= P\left[0_{(0,s_1)} 1_{(s_1,s_1+\Delta)} 0_{(s_1+\Delta,s_2)} 1_{(s_2,s_2+\Delta)} 0_{(s_2+\Delta,s_3)} \cdots 1_{(s_n,s_n+\Delta)} 0_{(s_n+\Delta,T)}\right] \tag{9}$$

The set of events $0_{(0,s_1)}$, $0_{(s_n+\Delta,T)}$, and for $i = 1, \ldots, n-1$, $0_{(s_i+\Delta,s_{i+1})}$ and $1_{(s_i,s_i+\Delta)}$ are independent because each devent depend on the Poisson process in a time interval that overlaps none of the other time intervals. In addition, since the Poisson process has rate λ, $P[0_{(s,t)}] = e^{-\lambda(t-s)}$ and $P[1_{(s_i,s_i+\Delta)}] = (\lambda\Delta)e^{-\lambda\Delta}$. Thus,

$$\begin{aligned}
&P\left[s_1 < S_1 \leq s_1 + \Delta, \ldots, s_n < S_n \leq s_n + \Delta, N = n\right] \\
&\quad = P\left[0_{(0,s_1)}\right] P\left[1_{(s_1,s_1+\Delta)}\right] P\left[0_{(s_1+\Delta,s_2)}\right] \cdots P\left[1_{(s_n,s_n+\Delta)}\right] P\left[0_{(s_n+\Delta,T)}\right] && (10) \\
&\quad = e^{-\lambda s_1}\left(\lambda\Delta e^{-\lambda\Delta}\right) e^{-\lambda(s_2-s_1-\Delta)} \cdots \left(\lambda\Delta e^{-\lambda\Delta}\right) e^{-\lambda(T-s_n-\Delta)} && (11) \\
&\quad = (\lambda\Delta)^n e^{-\lambda T} && (12)
\end{aligned}$$