ECE 543
Stochastic Signals and Systems
PRoblem Set 3 Solution

Problem Solutions : Yates and Goodman, 4.1.6 4.2.8 4.4.3 4.6.8 4.8.6 4.9.14 4.10.17 5.1.3 5.4.7 5.5.1 5.5.4 5.6.9 5.7.6 and 5.7.7

Problem 4.1.6 Solution

The given function is

$$F_{X,Y}(x, y) = \begin{cases} 1 - e^{-(x+y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we find the CDF $F_X(x)$ and $F_Y(y)$.

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Hence, for any $x \geq 0$ or $y \geq 0$,

$$P[X > x] = 0 \qquad P[Y > y] = 0 \tag{4}$$

For $x \geq 0$ and $y \geq 0$, this implies

$$P[\{X > x\} \cup \{Y > y\}] \leq P[X > x] + P[Y > y] = 0 \tag{5}$$

However,

$$P[\{X > x\} \cup \{Y > y\}] = 1 - P[X \leq x, Y \leq y] = 1 - (1 - e^{-(x+y)}) = e^{-(x+y)} \tag{6}$$

Thus, we have the contradiction that $e^{-(x+y)} \leq 0$ for all $x, y \geq 0$. We can conclude that the given function is not a valid CDF.

Problem 4.2.8 Solution

Each circuit test produces an acceptable circuit with probability p. Let K denote the number of rejected circuits that occur in n tests and X is the number of acceptable circuits before the first reject. The joint PMF, $P_{K,X}(k, x) = P[K = k, X = x]$ can be found by realizing that $\{K = k, X = x\}$ occurs if and only if the following events occur:

A The first x tests must be acceptable.

B Test $x + 1$ must be a rejection since otherwise we would have $x + 1$ acceptable at the beginnning.

C The remaining $n - x - 1$ tests must contain $k - 1$ rejections.

Since the events A, B and C are independent, the joint PMF for $x+k \le r$, $x \ge 0$ and $k \ge 0$ is

$$P_{K,X}(k,x) = \underbrace{p^x}_{P[A]} \underbrace{(1-p)}_{P[B]} \underbrace{\binom{n-x-1}{k-1}(1-p)^{k-1}p^{n-x-1-(k-1)}}_{P[C]} \tag{1}$$

After simplifying, a complete expression for the joint PMF is

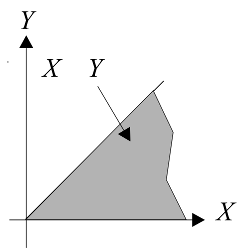
$$P_{K,X}(k,x) = \begin{cases} \binom{n-x-1}{k-1} p^{n-k}(1-p)^k & x+k \le n, x \ge 0, k \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.4.3 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \ge 0, y \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The probability that $X \ge Y$ is:

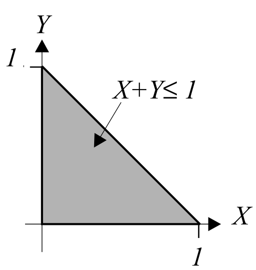


$$P[X \ge Y] = \int_0^\infty \int_0^x 6e^{-(2x+3y)}\, dy\, dx \tag{2}$$

$$= \int_0^\infty 2e^{-2x} \left(-e^{-3y}\Big|_{y=0}^{y=x} \right) dx \tag{3}$$

$$= \int_0^\infty [2e^{-2x} - 2e^{-5x}]\, dx = 3/5 \tag{4}$$

The $P[X+Y \le 1]$ is found by integrating over the region where $X+Y \le 1$



$$P[X+Y \le 1] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx \tag{5}$$

$$= \int_0^1 2e^{-2x} \left[-e^{-3y}\Big|_{y=0}^{y=1-x} \right] dx \tag{6}$$

$$= \int_0^1 2e^{-2x} \left[1 - e^{-3(1-x)} \right] dx \tag{7}$$

$$= -e^{-2x} - 2e^{x-3}\Big|_0^1 \tag{8}$$

$$= 1 + 2e^{-3} - 3e^{-2} \tag{9}$$

(b) The event $\min(X,Y) \ge 1$ is the same as the event $\{X \ge 1, Y \ge 1\}$. Thus,

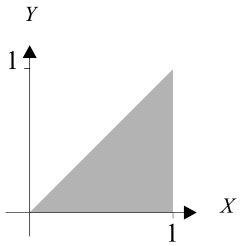
$$P[\min(X,Y) \ge 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)}\, dy\, dx = e^{-(2+3)} \tag{10}$$

(c) The event $\max(X,Y) \le 1$ is the same as the event $\{X \le 1, Y \le 1\}$ so that

$$P[\max(X,Y) \le 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)}\, dy\, dx = (1-e^{-2})(1-e^{-3}) \tag{11}$$

Problem 4.6.8 Solution

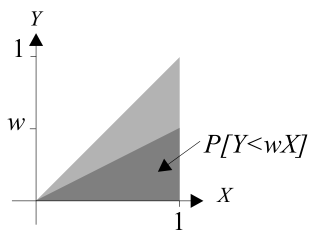
Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since X and Y are both nonnegative, $W = Y/X \ge 0$. Since $Y \le X$, $W = Y/X \le 1$. Note that $W = 0$ can occur if $Y = 0$. Thus the range of W is $S_W = \{w|0 \le w \le 1\}$.

(b) For $0 \le w \le 1$, the CDF of W is



$$F_W(w) = P[Y/X \le w] = P[Y \le wX] = w \tag{2}$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w & 0 \le w < 1 \\ 1 & w \ge 1 \end{cases} \tag{3}$$

By taking the derivative of the CDF, we find that the PDF of W is

$$f_W(w) = \begin{cases} 1 & 0 \le w < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We see that W has a uniform PDF over $[0,1]$. Thus $E[W] = 1/2$.

Problem 4.8.6 Solution

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (4x+2y)/3 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The probability of event $A = \{Y \le 1/2\}$ is

$$P[A] = \iint_{y \le 1/2} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^{1/2} \frac{4x+2y}{3}\, dy\, dx. \tag{2}$$

With some calculus,

$$P[A] = \int_0^1 \left. \frac{4xy + y^2}{3} \right|_{y=0}^{y=1/2} dx = \int_0^1 \frac{2x + 1/4}{3}\, dx = \left. \frac{x^2}{3} + \frac{x}{12} \right|_0^1 = \frac{5}{12}. \tag{3}$$

(b) The conditional joint PDF of X and Y given A is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$= \begin{cases} 8(2x+y)/5 & 0 \le x \le 1, 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

For $0 \le x \le 1$, the PDF of X given A is

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\; dy = \frac{8}{5}\int_0^{1/2} (2x+y)\, dy \tag{6}$$

$$= \frac{8}{5}\left(2xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1/2} = \frac{8x+1}{5} \tag{7}$$

The complete expression is

$$f_{X|A}(x) = \begin{cases} (8x+1)/5 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

For $0 \le y \le 1/2$, the conditional marginal PDF of Y given A is

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\; dx = \frac{8}{5}\int_0^{1} (2x+y)\, dx \tag{9}$$

$$= \frac{8x^2+8xy}{5}\Bigg|_{x=0}^{x=1} = \frac{8y+8}{5} \tag{10}$$

The complete expression is

$$f_{Y|A}(y) = \begin{cases} (8y+8)/5 & 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Problem 4.9.14 Solution

(a) The number of buses, N, must be greater than zero. Also, the number of minutes that pass cannot be less than the number of buses. Thus, $P[N = n, T = t] > 0$ for integers n, t satisfying $1 \le n \le t$.

(b) First, we find the joint PMF of N and T by carefully considering the possible sample paths. In particular, $P_{N,T}(n,t) = P[ABC] = P[A]P[B]P[C]$ where the events A, B and C are

$$A = \{n-1 \text{ buses arrive in the first } t-1 \text{ minutes}\} \tag{1}$$

$$B = \{\text{none of the first } n-1 \text{ buses are boarded}\} \tag{2}$$

$$C = \{\text{at time } t \text{ a bus arrives and is boarded}\} \tag{3}$$

These events are independent since each trial to board a bus is independent of when the buses arrive. These events have probabilities

$$P[A] = \binom{t-1}{n-1} p^{n-1}(1-p)^{t-1-(n-1)} \tag{4}$$

$$P[B] = (1-q)^{n-1} \tag{5}$$

$$P[C] = pq \tag{6}$$

Consequently, the joint PMF of N and T is

$$P_{N,T}(n,t) = \begin{cases} \binom{t-1}{n-1} p^{n-1}(1-p)^{t-n}(1-q)^{n-1}pq & n \geq 1, t \geq n \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(c) It is possible to find the marginal PMF's by summing the joint PMF. However, it is much easier to obtain the marginal PMFs by consideration of the experiment. Specifically, when a bus arrives, it is boarded with probability q. Moreover, the experiment ends when a bus is boarded. By viewing whether each arriving bus is boarded as an independent trial, N is the number of trials until the first success. Thus, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-q)^{n-1}q & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

To find the PMF of T, suppose we regard each minute as an independent trial in which a success occurs if a bus arrives and that bus is boarded. In this case, the success probability is pq and T is the number of minutes up to and including the first success. The PMF of T is also geometric.

$$P_T(t) = \begin{cases} (1-pq)^{t-1}pq & t = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

(d) Once we have the marginal PMFs, the conditional PMFs are easy to find.

$$P_{N|T}(n|t) = \frac{P_{N,T}(n,t)}{P_T(t)} = \begin{cases} \binom{t-1}{n-1} \left(\frac{p(1-q)}{1-pq}\right)^{n-1} \left(\frac{1-p}{1-pq}\right)^{t-1-(n-1)} & n = 1, 2, \ldots, t \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

That is, given you depart at time $T = t$, the number of buses that arrive during minutes $1, \ldots, t-1$ has a binomial PMF since in each minute a bus arrives with probability p. Similarly, the conditional PMF of T given N is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \binom{t-1}{n-1} p^n (1-p)^{t-n} & t = n, n+1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

This result can be explained. Given that you board bus $N = n$, the time T when you leave is the time for n buses to arrive. If we view each bus arrival as a success of an independent trial, the time for n buses to arrive has the above Pascal PMF.

Problem 4.10.17 Solution

We need to define the events $A = \{U \le u\}$ and $B = \{V \le v\}$. In this case,

$$F_{U,V}(u,v) = P[AB] = P[B] - P[A^c B] = P[V \le v] - P[U > u, V \le v] \tag{1}$$

Note that $U = \min(X,Y) > u$ if and only if $X > u$ and $Y > u$. In the same way, since $V = \max(X,Y)$, $V \le v$ if and only if $X \le v$ and $Y \le v$. Thus

$$\begin{aligned} P[U > u, V \le v] &= P[X > u, Y > u, X \le v, Y \le v] &(2)\\ &= P[u < X \le v, u < Y \le v] &(3)\end{aligned}$$

Thus, the joint CDF of U and V satisfies

$$\begin{aligned} F_{U,V}(u,v) &= P[V \le v] - P[U > u, V \le v] &(4)\\ &= P[X \le v, Y \le v] - P[u < X \le v, u < X \le v] &(5)\end{aligned}$$

Since X and Y are independent random variables,

$$\begin{aligned} F_{U,V}(u,v) &= P[X \le v]\, P[Y \le v] - P[u < X \le v]\, P[u < X \le v] &(6)\\ &= F_X(v)\, F_Y(v) - (F_X(v) - F_X(u))\,(F_Y(v) - F_Y(u)) &(7)\\ &= F_X(v)\, F_Y(u) + F_X(u)\, F_Y(v) - F_X(u)\, F_Y(u) &(8)\end{aligned}$$

The joint PDF is

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\partial^2 F_{U,V}(u,v)}{\partial u \partial v} &(9)\\ &= \frac{\partial}{\partial u}\left[f_X(v)\, F_Y(u) + F_X(u)\, f_Y(v)\right] &(10)\\ &= f_X(u)\, f_Y(v) + f_X(v)\, f_Y(v) &(11)\end{aligned}$$

Problem 5.1.3 Solution

(a) In terms of the joint PDF, we can write joint CDF as

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(y_1,\ldots,y_n)\, dy_1 \cdots dy_n \tag{1}$$

However, simplifying the above integral depends on the values of each x_i. In particular, $f_{X_1,\ldots,X_n}(y_1,\ldots,y_n) = 1$ if and only if $0 \le y_i \le 1$ for each i. Since $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = 0$ if any $x_i < 0$, we limit, for the moment, our attention to the case where $x_i \ge 0$ for all i. In this case, some thought will show that we can write the limits in the following way:

$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) &= \int_0^{\max(1,x_1)} \cdots \int_0^{\min(1,x_n)} dy_1 \cdots dy_n &(2)\\ &= \min(1,x_1)\min(1,x_2)\cdots\min(1,x_n) &(3)\end{aligned}$$

A complete expression for the CDF of $X_1,\ldots,X_n$ is

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^n \min(1,x_i) & 0 \le x_i, i = 1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) For $n = 3$,

$$1 - P\left[\min_i X_i \le 3/4\right] = P\left[\min_i X_i > 3/4\right] \tag{5}$$
$$= P\left[X_1 > 3/4, X_2 > 3/4, X_3 > 3/4\right] \tag{6}$$
$$= \int_{3/4}^{1}\int_{3/4}^{1}\int_{3/4}^{1} dx_1\, dx_2\, dx_3 \tag{7}$$
$$= (1 - 3/4)^3 = 1/64 \tag{8}$$

Thus $P[\min_i X_i \le 3/4] = 63/64$.

Problem 5.4.7 Solution

Since $U_1, \ldots, U_n$ are iid uniform $(0, 1)$ random variables,

$$f_{U_1,\ldots,U_n}(u_1, \ldots, u_n) = \begin{cases} 1/T^n & 0 \le u_i \le 1; i = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since $U_1, \ldots, U_n$ are continuous, $P[U_i = U_j] = 0$ for all $i \ne j$. For the same reason, $P[X_i = X_j] = 0$ for $i \ne j$. Thus we need only to consider the case when $x_1 < x_2 < \cdots < x_n$.

To understand the claim, it is instructive to start with the $n = 2$ case. In this case, $(X_1, X_2) = (x_1, x_2)$ (with $x_1 < x_2$) if either $(U_1, U_2) = (x_1, x_2)$ or $(U_1, U_2) = (x_2, x_1)$. For infinitesimal Δ,

$$f_{X_1,X_2}(x_1, x_2)\,\Delta^2 = P\left[x_1 < X_1 \le x_1 + \Delta, x_2 < X_2 \le x_2 + \Delta\right] \tag{2}$$
$$= P\left[x_1 < U_1 \le x_1 + \Delta, x_2 < U_2 \le x_2 + \Delta\right] + P\left[x_2 < U_1 \le x_2 + \Delta, x_1 < U_2 \le x_1 + \Delta\right] \tag{3}$$
$$= f_{U_1,U_2}(x_1, x_2)\,\Delta^2 + f_{U_1,U_2}(x_2, x_1)\,\Delta^2 \tag{4}$$

We see that for $0 \le x_1 < x_2 \le 1$ that

$$f_{X_1,X_2}(x_1, x_2) = 2/T^n. \tag{5}$$

For the general case of n uniform random variables, we define $\boldsymbol{\pi} = \begin{bmatrix}\pi(1) & \ldots & \pi(n)\end{bmatrix}'$ as a permutation vector of the integers $1, 2, \ldots, n$ and Π as the set of $n!$ possible permutation vectors. In this case, the event $\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\}$ occurs if

$$U_1 = x_{\pi(1)}, U_2 = x_{\pi(2)}, \ldots, U_n = x_{\pi(n)} \tag{6}$$

for any permutation $\boldsymbol{\pi} \in \Pi$. Thus, for $0 \le x_1 < x_2 < \cdots < x_n \le 1$,

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\,\Delta^n = \sum_{\boldsymbol{\pi}\in\Pi} f_{U_1,\ldots,U_n}\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)\Delta^n. \tag{7}$$

Since there are $n!$ permutations and $f_{U_1,\ldots,U_n}(x_{\pi(1)}, \ldots, x_{\pi(n)}) = 1/T^n$ for each permutation $\boldsymbol{\pi}$, we can conclude that

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = n!/T^n. \tag{8}$$

Since the order statistics are necessarily ordered, $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = 0$ unless $x_1 < \cdots < x_n$.

Problem 5.5.1 Solution
For discrete random vectors, it is true in general that

$$P_{\mathbf{Y}}(\mathbf{y}) = P[\mathbf{Y} = \mathbf{y}] = P[\mathbf{AX} + \mathbf{b} = \mathbf{y}] = P[\mathbf{AX} = \mathbf{y} - \mathbf{b}]. \tag{1}$$

For an arbitrary matrix $\mathbf{A}$, the system of equations $\mathbf{Ax} = \mathbf{y} - \mathbf{b}$ may have no solutions (if the columns of $\mathbf{A}$ do not span the vector space), multiple solutions (if the columns of $\mathbf{A}$ are linearly dependent), or, when $\mathbf{A}$ is invertible, exactly one solution. In the invertible case,

$$P_{\mathbf{Y}}(\mathbf{y}) = P[\mathbf{AX} = \mathbf{y} - \mathbf{b}] = P\left[\mathbf{X} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})\right] = P_{\mathbf{X}}\left(\mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})\right). \tag{2}$$

As an aside, we note that when $\mathbf{Ax} = \mathbf{y} - \mathbf{b}$ has multiple solutions, we would need to do some bookkeeping to add up the probabilities $P_{\mathbf{X}}(\mathbf{x})$ for all vectors $\mathbf{x}$ satisfying $\mathbf{Ax} = \mathbf{y} - \mathbf{b}$. This can get disagreeably complicated.

Problem 5.5.4 Solution
Let X_i denote the finishing time of boat i. Since finishing times of all boats are iid Gaussian random variables with expected value 35 minutes and standard deviation 5 minutes, we know that each X_i has CDF

$$F_{X_i}(x) = P[X_i \le x] = P\left[\frac{X_i - 35}{5} \le \frac{x - 35}{5}\right] = \Phi\left(\frac{x - 35}{5}\right) \tag{1}$$

(a) The time of the winning boat is

$$W = \min(X_1, X_2, \ldots, X_{10}) \tag{2}$$

To find the probability that $W \le 25$, we will find the CDF $F_W(w)$ since this will also be useful for part (c).

$$\begin{aligned} F_W(w) &= P[\min(X_1, X_2, \ldots, X_{10}) \le w] && (3)\\ &= 1 - P[\min(X_1, X_2, \ldots, X_{10}) > w] && (4)\\ &= 1 - P[X_1 > w, X_2 > w, \ldots, X_{10} > w] && (5) \end{aligned}$$

Since the X_i are iid,

$$\begin{aligned} F_W(w) = 1 - \prod_{i=1}^{10} P[X_i > w] &= 1 - (1 - F_{X_i}(w))^{10} && (6)\\ &= 1 - \left(1 - \Phi\left(\frac{w - 35}{5}\right)\right)^{10} && (7) \end{aligned}$$

Thus,

$$\begin{aligned} P[W \le 25] = F_W(25) &= 1 - (1 - \Phi(-2))^{10} && (8)\\ &= 1 - [\Phi(2)]^{10} = 0.2056. && (9) \end{aligned}$$

(b) The finishing time of the last boat is $L = \max(X_1, \ldots, X_{10})$. The probability that the last boat finishes in more than 50 minutes is

$$P[L > 50] = 1 - P[L \leq 50] \tag{10}$$
$$= 1 - P[X_1 \leq 50, X_2 \leq 50, \ldots, X_{10} \leq 50] \tag{11}$$

Once again, since the X_i are iid Gaussian $(35, 5)$ random variables,

$$P[L > 50] = 1 - \prod_{i=1}^{10} P[X_i \leq 50] = 1 - (F_{X_i}(50))^{10} \tag{12}$$
$$= 1 - (\Phi([50 - 35]/5))^{10} \tag{13}$$
$$= 1 - (\Phi(3))^{10} = 0.0134 \tag{14}$$

(c) A boat will finish in negative time if and only iff the winning boat finishes in negative time, which has probability

$$F_W(0) = 1 - (1 - \Phi(-35/5))^{10} = 1 - (1 - \Phi(-7))^{10} = 1 - (\Phi(7))^{10}. \tag{15}$$

Unfortunately, the tables in the text have neither $\Phi(7)$ nor $Q(7)$. However, those with access to MATLAB, or a programmable calculator, can find out that $Q(7) = 1 - \Phi(7) = 1.28 \times 10^{-12}$. This implies that a boat finishes in negative time with probability

$$F_W(0) = 1 - (1 - 1.28 \times 10^{-12})^{10} = 1.28 \times 10^{-11}. \tag{16}$$

Problem 5.6.9 Solution

Given an arbitrary random vector $\mathbf{X}$, we can define $\mathbf{Y} = \mathbf{X} - \boldsymbol{\mu}_\mathbf{X}$ so that

$$\mathbf{C}_\mathbf{X} = E\left[(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})'\right] = E\left[\mathbf{Y}\mathbf{Y}'\right] = \mathbf{R}_\mathbf{Y}. \tag{1}$$

It follows that the covariance matrix $\mathbf{C}_\mathbf{X}$ is positive semi-definite if and only if the correlation matrix $\mathbf{R}_\mathbf{Y}$ is positive semi-definite. Thus, it is sufficient to show that every correlation matrix, whether it is denoted $\mathbf{R}_\mathbf{Y}$ or $\mathbf{R}_\mathbf{X}$, is positive semi-definite.

To show a correlation matrix $\mathbf{R}_\mathbf{X}$ is positive semi-definite, we write

$$\mathbf{a}'\mathbf{R}_\mathbf{X}\mathbf{a} = \mathbf{a}'E\left[\mathbf{X}\mathbf{X}'\right]\mathbf{a} = E\left[\mathbf{a}'\mathbf{X}\mathbf{X}'\mathbf{a}\right] = E\left[(\mathbf{a}'\mathbf{X})(\mathbf{X}'\mathbf{a})\right] = E\left[(\mathbf{a}'\mathbf{X})^2\right]. \tag{2}$$

We note that $W = \mathbf{a}'\mathbf{X}$ is a random variable. Since $E[W^2] \geq 0$ for any random variable W,

$$\mathbf{a}'\mathbf{R}_\mathbf{X}\mathbf{a} = E\left[W^2\right] \geq 0. \tag{3}$$

Problem 5.7.6 Solution

(a) From Theorem 5.13, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C_Y} = \mathbf{QC_XQ'} \tag{1}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta & (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta \\ (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta & \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta \end{bmatrix}. \tag{3}$$

We conclude that Y_1 and Y_2 have covariance

$$\text{Cov}[Y_1, Y_2] = C_{\mathbf{Y}}(1,2) = (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta. \tag{4}$$

Since Y_1 and Y_2 are jointly Gaussian, they are independent if and only if $\text{Cov}[Y_1, Y_2] = 0$. Thus, Y_1 and Y_2 are independent for all θ if and only if $\sigma_1^2 = \sigma_2^2$. In this case, when the joint PDF $f_{\mathbf{X}}(\mathbf{x})$ is symmetric in x_1 and x_2. In terms of polar coordinates, the PDF $f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,X_2}(x_1, x_2)$ depends on $r = \sqrt{x_1^2 + x_2^2}$ but for a given r, is constant for all $\phi = \tan^{-1}(x_2/x_1)$. The transformation of $\mathbf{X}$ to $\mathbf{Y}$ is just a rotation of the coordinate system by θ preserves this circular symmetry.

(b) If $\sigma_2^2 > \sigma_1^2$, then Y_1 and Y_2 are independent if and only if $\sin\theta\cos\theta = 0$. This occurs in the following cases:

- $\theta = 0$: $Y_1 = X_1$ and $Y_2 = X_2$
- $\theta = \pi/2$: $Y_1 = -X_2$ and $Y_2 = -X_1$
- $\theta = \pi$: $Y_1 = -X_1$ and $Y_2 = -X_2$
- $\theta = -\pi/2$: $Y_1 = X_2$ and $Y_2 = X_1$

In all four cases, Y_1 and Y_2 are just relabeled versions, possibly with sign changes, of X_1 and X_2. In these cases, Y_1 and Y_2 are independent because X_1 and X_2 are independent. For other values of θ, each Y_i is a linear combination of both X_1 and X_2. This mixing results in correlation between Y_1 and Y_2.

Problem 5.7.7 Solution

The difficulty of this problem is overrated since its a pretty simple application of Problem 5.7.6. In particular,

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\Bigg|_{\theta=45^\circ} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

Since $\mathbf{X} = \mathbf{QY}$, we know from Theorem 5.16 that $\mathbf{X}$ is Gaussian with covariance matrix

$$\mathbf{C_X} = \mathbf{QC_YQ'} \tag{2}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{3}$$

$$= \frac{1}{2} \begin{bmatrix} 1+\rho & -(1-\rho) \\ 1+\rho & 1-\rho \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{5}$$