**ECE 541**
**Stochastic Signals and Systems**
**Problem Set 11 Solution**

**Problem Solutions** : Yates and Goodman, 11.1.4 11.2.7 11.3.3 11.4.3 11.8.3 and 11.8.10

## Problem 11.1.4 Solution

Since $E[Y^2(t)] = R_Y(0)$, we use Theorem 11.2(a) to evaluate $R_Y(\tau)$ at $\tau = 0$. That is,

$$\begin{aligned} R_Y(0) &= \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(u-v)\, dv\, du && (1) \\ &= \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v)\eta_0 \delta(u-v)\, dv\, du && (2) \\ &= \eta_0 \int_{-\infty}^{\infty} h^2(u)\, du, && (3) \end{aligned}$$

by the sifting property of the delta function.

## Problem 11.2.7 Solution

There is a technical difficulty with this problem since $X_n$ is not defined for $n < 0$. This implies $C_X[n,k]$ is not defined for $k < -n$ and thus $C_X[n,k]$ cannot be completely independent of $k$. When $n$ is large, corresponding to a process that has been running for a long time, this is a technical issue, and not a practical concern. Instead, we will find $\bar{\sigma}^2$ such that $C_X[n,k] = C_X[k]$ for all $n$ and $k$ for which the covariance function is defined. To do so, we need to express $X_n$ in terms of $Z_0, Z_1, \ldots, Z_{n_1}$. We do this in the following way:

$$\begin{aligned} X_n &= cX_{n-1} + Z_{n-1} && (1) \\ &= c[cX_{n-2} + Z_{n-2}] + Z_{n-1} && (2) \\ &= c^2[cX_{n-3} + Z_{n-3}] + cZ_{n-2} + Z_{n-1} && (3) \\ &\vdots && (4) \\ &= c^n X_0 + c^{n-1} Z_0 + c^{n-2} Z_2 + \cdots + Z_{n-1} && (5) \\ &= c^n X_0 + \sum_{i=0}^{n-1} c^{n-1-i} Z_i && (6) \end{aligned}$$

Since $E[Z_i] = 0$, the mean function of the $X_n$ process is

$$E[X_n] = c^n E[X_0] + \sum_{i=0}^{n-1} c^{n-1-i} E[Z_i] = E[X_0] \quad (7)$$

Thus, for $X_n$ to be a zero mean process, we require that $E[X_0] = 0$. The autocorrelation function can be written as

$$R_X[n,k] = E[X_n X_{n+k}] = E\left[\left(c^n X_0 + \sum_{i=0}^{n-1} c^{n-1-i} Z_i\right)\left(c^{n+k} X_0 + \sum_{j=0}^{n+k-1} c^{n+k-1-j} Z_j\right)\right] \quad (8)$$

Although it was unstated in the problem, we will assume that $X_0$ is independent of $Z_0, Z_1, \ldots$ so that $E[X_0 Z_i] = 0$. Since $E[Z_i] = 0$ and $E[Z_i Z_j] = 0$ for $i \neq j$, most of the cross terms will drop out. For $k \geq 0$, autocorrelation simplifies to

$$R_X[n,k] = c^{2n+k}\operatorname{Var}[X_0] + \sum_{i=0}^{n-1} c^{2(n-1)+k-2i}\bar{\sigma}^2 = c^{2n+k}\operatorname{Var}[X_0] + \bar{\sigma}^2 c^k \frac{1-c^{2n}}{1-c^2} \tag{9}$$

Since $E[X_n] = 0$, $\operatorname{Var}[X_0] = R_X[n,0] = \sigma^2$ and we can write for $k \geq 0$,

$$R_X[n,k] = \bar{\sigma}^2 \frac{c^k}{1-c^2} + c^{2n+k}\left(\sigma^2 - \frac{\bar{\sigma}^2}{1-c^2}\right) \tag{10}$$

For $k < 0$, we have

$$R_X[n,k] = E\left[\left(c^n X_0 + \sum_{i=0}^{n-1} c^{n-1-i} Z_i\right)\left(c^{n+k} X_0 + \sum_{j=0}^{n+k-1} c^{n+k-1-j} Z_j\right)\right] \tag{11}$$

$$= c^{2n+k}\operatorname{Var}[X_0] + c^{-k}\sum_{j=0}^{n+k-1} c^{2(n+k-1-j)}\bar{\sigma}^2 \tag{12}$$

$$= c^{2n+k}\sigma^2 + \bar{\sigma}^2 c^{-k}\frac{1-c^{2(n+k)}}{1-c^2} \tag{13}$$

$$= \frac{\bar{\sigma}^2}{1-c^2}c^{-k} + c^{2n+k}\left(\sigma^2 - \frac{\bar{\sigma}^2}{1-c^2}\right) \tag{14}$$

We see that $R_X[n,k] = \sigma^2 c^{|k|}$ by choosing

$$\bar{\sigma}^2 = (1-c^2)\sigma^2 \tag{15}$$

## Problem 11.3.3 Solution

The sequence $X_n$ is passed through the filter

$$\mathbf{h} = \begin{bmatrix} h_0 & h_1 & h_2 \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}' \tag{1}$$

The output sequence is $Y_n$. Following the approach of Equation (11.58), we can write the output $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}'$ as

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} X_{-1} \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} X_{-1} \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}}_{\mathbf{X}}. \tag{2}$$

Since $X_n$ has autocovariance function $C_X(k) = 2^{-|k|}$, $\mathbf{X}$ has covariance matrix

$$\mathbf{C_X} = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 & 1/16 \\ 1/2 & 1 & 1/2 & 1/4 & 1/8 \\ 1/4 & 1/2 & 1 & 1/2 & 1/4 \\ 1/8 & 1/4 & 1/2 & 1 & 1/2 \\ 1/16 & 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}. \tag{3}$$

Since $\mathbf{Y} = \mathbf{H}\mathbf{X}$,

$$\mathbf{C_Y} = \mathbf{H}\mathbf{C_X}\mathbf{H}' = \begin{bmatrix} 3/2 & -3/8 & 9/16 \\ -3/8 & 3/2 & -3/8 \\ 9/16 & -3/8 & 3/2 \end{bmatrix}. \tag{4}$$

Some calculation (by hand or preferably by MATLAB) will show that $\det(\mathbf{C_Y}) = 675/256$ and that

$$\mathbf{C_Y}^{-1} = \frac{1}{15}\begin{bmatrix} 12 & 2 & -4 \\ 2 & 11 & 2 \\ -4 & 2 & 12 \end{bmatrix}. \tag{5}$$

Some algebra will show that

$$\mathbf{y}'\mathbf{C_Y}^{-1}\mathbf{y} = \frac{12y_1^2 + 11y_2^2 + 12y_3^2 + 4y_1y_2 + -8y_1y_3 + 4y_2y_3}{15}. \tag{6}$$

This implies $\mathbf{Y}$ has PDF

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{3/2}[\det(\mathbf{C_Y})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{y}'\mathbf{C_Y}^{-1}\mathbf{y}\right) \tag{7}$$

$$= \frac{16}{(2\pi)^{3/2}15\sqrt{3}} \exp\left(-\frac{12y_1^2 + 11y_2^2 + 12y_3^2 + 4y_1y_2 + -8y_1y_3 + 4y_2y_3}{30}\right). \tag{8}$$

This solution is another demonstration of why the PDF of a Gaussian random vector should be left in vector form.

**Comment:** We know from Theorem 11.5 that $Y_n$ is a stationary Gaussian process. As a result, the random variables $Y_1$, $Y_2$ and $Y_3$ are identically distributed and $\mathbf{C_Y}$ is a symmetric Toeplitz matrix. This might make on think that the PDF $f_{\mathbf{Y}}(\mathbf{y})$ should be symmetric in the variables $y_1$, $y_2$ and $y_3$. However, because $Y_2$ is in the middle of $Y_1$ and $Y_3$, the information provided by $Y_1$ and $Y_3$ about $Y_2$ is different than the information $Y_1$ and $Y_2$ convey about $Y_3$. This fact appears as asymmetry in $f_{\mathbf{Y}}(\mathbf{y})$.

## Problem 11.4.3 Solution

This problem generalizes Example 11.14 in that $-0.9$ is replaced by the parameter $c$ and the noise variance 0.2 is replaced by $\eta^2$. Because we are only finding the first order filter $\mathbf{h} = \begin{bmatrix} h_0 & h_1 \end{bmatrix}'$, it is relatively simple to generalize the solution of Example 11.14 to the parameter values $c$ and $\eta^2$.

Based on the observation $\mathbf{Y} = \begin{bmatrix} Y_{n-1} & Y_n \end{bmatrix}'$, Theorem 11.11 states that the linear MMSE estimate of $X = X_n$ is $\overleftarrow{\mathbf{h}}'\mathbf{Y}$ where

$$\overleftarrow{\mathbf{h}} = \mathbf{R_Y}^{-1}\mathbf{R}_{\mathbf{Y}X_n} = (\mathbf{R}_{\mathbf{X}_n} + \mathbf{R}_{\mathbf{W}_n})^{-1}\mathbf{R}_{\mathbf{X}_n X_n}. \tag{1}$$

From Equation (11.82), $\mathbf{R}_{\mathbf{X}_n X_n} = \begin{bmatrix} R_X[1] & R_X[0] \end{bmatrix}' = \begin{bmatrix} c & 1 \end{bmatrix}'$. From the problem statement,

$$\mathbf{R}_{\mathbf{X}_n} + \mathbf{R}_{\mathbf{W}_n} = \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} + \begin{bmatrix} \eta^2 & 0 \\ 0 & \eta^2 \end{bmatrix} = \begin{bmatrix} 1+\eta^2 & c \\ c & 1+\eta^2 \end{bmatrix}. \tag{2}$$

This implies

$$\overleftarrow{\mathbf{h}} = \begin{bmatrix} 1+\eta^2 & c \\ c & 1+\eta^2 \end{bmatrix}^{-1} \begin{bmatrix} c \\ 1 \end{bmatrix} \tag{3}$$

$$= \frac{1}{(1+\eta^2)^2 - c^2} \begin{bmatrix} 1+\eta^2 & -c \\ -c & 1+\eta^2 \end{bmatrix} \begin{bmatrix} c \\ 1 \end{bmatrix} \tag{4}$$

$$= \frac{1}{(1+\eta^2)^2 - c^2} \begin{bmatrix} c\eta^2 \\ 1+\eta^2 - c^2 \end{bmatrix}. \tag{5}$$

The optimal filter is

$$\mathbf{h} = \frac{1}{(1+\eta^2)^2 - c^2} \begin{bmatrix} 1+\eta^2 - c^2 \\ c\eta^2 \end{bmatrix}. \tag{6}$$

To find the mean square error of this predictor, we recall that Theorem 11.11 is just Theorem 9.7 expressed in the terminology of filters. Expressing part (c) of Theorem 9.7 in terms of the linear estimation filter $\mathbf{h}$, the mean square error of the estimator is

$$e_L^* = \text{Var}[X_n] - \overleftarrow{\mathbf{h}}' \mathbf{R}_{\mathbf{Y}_n X_n} \tag{7}$$

$$= \text{Var}[X_n] - \overleftarrow{\mathbf{h}}' \mathbf{R}_{\mathbf{X}_n X_n} \tag{8}$$

$$= R_X[0] - \overleftarrow{\mathbf{h}}' \begin{bmatrix} c \\ 1 \end{bmatrix} \tag{9}$$

$$= 1 - \frac{c^2\eta^2 + \eta^2 + 1 - c^2}{(1+\eta^2)^2 - c^2}. \tag{10}$$

Note that we always find that $e_L^* < \text{Var}[X_n] = 1$ simply because the optimal estimator cannot be worse than the blind estimator that ignores the observation $\mathbf{Y}_n$.

## Problem 11.8.3 Solution

Since $S_Y(f) = |H(f)|^2 S_X(f)$, we first find

$$|H(f)|^2 = H(f)H^*(f) \tag{1}$$

$$= \left(a_1 e^{-j2\pi f t_1} + a_2 e^{-j2\pi f t_2}\right)\left(a_1 e^{j2\pi f t_1} + a_2 e^{j2\pi f t_2}\right) \tag{2}$$

$$= a_1^2 + a_2^2 + a_1 a_2 \left(e^{-j2\pi f(t_2 - t_1)} + e^{-j2\pi f(t_1 - t_2)}\right) \tag{3}$$

It follows that the output power spectral density is

$$S_Y(f) = (a_1^2 + a_2^2) S_X(f) + a_1 a_2 S_X(f) e^{-j2\pi f(t_2 - t_1)} + a_1 a_2 S_X(f) e^{-j2\pi f(t_1 - t_2)} \tag{4}$$

Using Table 11.1, the autocorrelation of the output is

$$R_Y(\tau) = (a_1^2 + a_2^2) R_X(\tau) + a_1 a_2 \left(R_X(\tau - (t_1 - t_2)) + R_X(\tau + (t_1 - t_2))\right) \tag{5}$$

## Problem 11.8.10 Solution

(a) Since $S_W(f) = 10^{-15}$ for all $f$, $R_W(\tau) = 10^{-15}\delta(\tau)$.

(b) Since $\Theta$ is independent of $W(t)$,

$$E[V(t)] = E[W(t)\cos(2\pi f_c t + \Theta)] = E[W(t)]\,E[\cos(2\pi f_c t + \Theta)] = 0 \tag{1}$$

(c) We cannot initially assume $V(t)$ is WSS so we first find

$$\begin{aligned}
R_V(t,\tau) &= E[V(t)V(t+\tau)] && (2)\\
&= E[W(t)\cos(2\pi f_c t+\Theta)W(t+\tau)\cos(2\pi f_c(t+\tau)+\Theta)] && (3)\\
&= E[W(t)W(t+\tau)]E[\cos(2\pi f_c t+\Theta)\cos(2\pi f_c(t+\tau)+\Theta)] && (4)\\
&= 10^{-15}\delta(\tau)E[\cos(2\pi f_c t+\Theta)\cos(2\pi f_c(t+\tau)+\Theta)] && (5)
\end{aligned}$$

We see that for all $\tau \neq 0$, $R_V(t, t+\tau) = 0$. Thus we need to find the expected value of

$$E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c(t+\tau) + \Theta)] \tag{6}$$

only at $\tau = 0$. However, its good practice to solve for arbitrary $\tau$:

$$\begin{aligned}
&E[\cos(2\pi f_c t+\Theta)\cos(2\pi f_c(t+\tau)+\Theta)] && (7)\\
&= \frac{1}{2}E[\cos(2\pi f_c\tau)+\cos(2\pi f_c(2t+\tau)+2\Theta)] && (8)\\
&= \frac{1}{2}\cos(2\pi f_c\tau)+\frac{1}{2}\int_0^{2\pi}\cos(2\pi f_c(2t+\tau)+2\theta)\frac{1}{2\pi}\,d\theta && (9)\\
&= \frac{1}{2}\cos(2\pi f_c\tau)+\frac{1}{2}\sin(2\pi f_c(2t+\tau)+2\theta)\Big|_0^{2\pi} && (10)\\
&= \frac{1}{2}\cos(2\pi f_c\tau)+\frac{1}{2}\sin(2\pi f_c(2t+\tau)+4\pi)-\frac{1}{2}\sin(2\pi f_c(2t+\tau)) && (11)\\
&= \frac{1}{2}\cos(2\pi f_c\tau) && (12)
\end{aligned}$$

Consequently,

$$R_V(t,\tau) = \frac{1}{2}10^{-15}\delta(\tau)\cos(2\pi f_c\tau) = \frac{1}{2}10^{-15}\delta(\tau) \tag{13}$$

(d) Since $E[V(t)] = 0$ and since $R_V(t,\tau) = R_V(\tau)$, we see that $V(t)$ is a wide sense stationary process. Since $L(f)$ is a linear time invariant filter, the filter output $Y(t)$ is also a wide sense stationary process.

(e) The filter input $V(t)$ has power spectral density $S_V(f) = \frac{1}{2}10^{-15}$. The filter output has power spectral density

$$S_Y(f) = |L(f)|^2 S_V(f) = \begin{cases} 10^{-15}/2 & |f| \le B \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

The average power of $Y(t)$ is

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} S_Y(f)\, df = \int_{-B}^{B} \frac{1}{2} 10^{-15}\, df = 10^{-15} B \tag{15}$$