Stochastic Signals and Systems
Problem Set 10 Solution

Problem Solutions : Yates and Goodman, 10.7.3 10.8.1 10.8.3 10.9.4 10.10.3 10.11.2 and 10.12.1

Problem 10.7.3 Solution

First we observe that $Y_n = X_n - X_{n-1} = X(n) - X(n-1)$ is a Gaussian random variable with mean zero and variance α. Since this fact is true for all n, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. By Definition 10.10 for Brownian motion, $Y_n = X(n) - X(n-1)$ is independent of $X(m)$ for any $m \leq n-1$. Hence Y_n is independent of $Y_m = X(m) - X(m-1)$ for any $m \leq n-1$. Equivalently, $Y_1, Y_2, \ldots$ is a sequence of independent random variables.

Problem 10.8.1 Solution

The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)] \tag{1}$$

for $k = 0$, $C_X[m,0] = \text{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]\,E[(X_{m+k} - \mu_X)] = 0 \tag{2}$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{3}$$

Problem 10.8.3 Solution

In this problem, the daily temperature process results from

$$C_n = 16\left[1 - \cos\frac{2\pi n}{365}\right] + 4X_n \tag{1}$$

where X_n is an iid random sequence of $N[0,1]$ random variables. The hardest part of this problem is distinguishing between the process C_n and the covariance function $C_C[k]$.

(a) The expected value of the process is

$$E[C_n] = 16E\left[1 - \cos\frac{2\pi n}{365}\right] + 4E[X_n] = 16\left[1 - \cos\frac{2\pi n}{365}\right] \tag{2}$$

(b) The autocovariance of C_n is

$$C_C[m,k] = E\left[\left(C_m - 16\left[1 - \cos\frac{2\pi m}{365}\right]\right)\left(C_{m+k} - 16\left[1 - \cos\frac{2\pi(m+k)}{365}\right]\right)\right] \tag{3}$$

$$= 16E[X_m X_{m+k}] = \begin{cases} 16 & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(c) A model of this type may be able to capture the mean and variance of the daily temperature. However, one reason this model is overly simple is because day to day temperatures are uncorrelated. A more realistic model might incorporate the effects of "heat waves" or "cold spells" through correlated daily temperatures.

Problem 10.9.4 Solution

Since $Y_n = X_{kn}$,

$$f_{Y_{n_1+l},\ldots,Y_{n_m+l}}(y_1,\ldots,y_m) = f_{X_{kn_1+kl},\ldots,X_{kn_m+kl}}(y_1,\ldots,y_m) \tag{1}$$

Stationarity of the X_n process implies

$$\begin{aligned} f_{X_{kn_1+kl},\ldots,X_{kn_m+kl}}(y_1,\ldots,y_m) &= f_{X_{kn_1},\ldots,X_{kn_m}}(y_1,\ldots,y_m) && (2)\\ &= f_{Y_{n_1},\ldots,Y_{n_m}}(y_1,\ldots,y_m). && (3) \end{aligned}$$

We combine these steps to write

$$f_{Y_{n_1+l},\ldots,Y_{n_m+l}}(y_1,\ldots,y_m) = f_{Y_{n_1},\ldots,Y_{n_m}}(y_1,\ldots,y_m). \tag{4}$$

Thus Y_n is a stationary process.

Comment: The first printing of the text asks whether Y_n is wide stationary if X_n is wide sense stationary. This fact is also true; however, since wide sense stationarity isn't addressed until the next section, the problem was corrected to ask about stationarity.

Problem 10.10.3 Solution

In this problem, we find the autocorrelation $R_W(t,\tau)$ when

$$W(t) = X\cos 2\pi f_0 t + Y\sin 2\pi f_0 t, \tag{1}$$

and X and Y are uncorrelated random variables with $E[X] = E[Y] = 0$.

We start by writing

$$\begin{aligned} R_W(t,\tau) &= E\left[W(t)W(t+\tau)\right] && (2)\\ &= E\left[(X\cos 2\pi f_0 t + Y\sin 2\pi f_0 t)\left(X\cos 2\pi f_0(t+\tau) + Y\sin 2\pi f_0(t+\tau)\right)\right]. && (3) \end{aligned}$$

Since X and Y are uncorrelated, $E[XY] = E[X]E[Y] = 0$. Thus, when we expand $E[W(t)W(t+\tau)]$ and take the expectation, all of the XY cross terms will be zero. This implies

$$R_W(t,\tau) = E\left[X^2\right]\cos 2\pi f_0 t\cos 2\pi f_0(t+\tau) + E\left[Y^2\right]\sin 2\pi f_0 t\sin 2\pi f_0(t+\tau) \tag{4}$$

Since $E[X] = E[Y] = 0$,

$$E\left[X^2\right] = \mathrm{Var}[X] - (E\left[X\right])^2 = \sigma^2, \qquad E\left[Y^2\right] = \mathrm{Var}[Y] - (E\left[Y\right])^2 = \sigma^2. \tag{5}$$

In addition, from Math Fact B.2, we use the formulas

$$\cos A \cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big] \tag{6}$$
$$\sin A \sin B = \frac{1}{2}\big[\cos(A-B) - \cos(A+B)\big] \tag{7}$$

to write

$$R_W(t,\tau) = \frac{\sigma^2}{2}\left(\cos 2\pi f_0\tau + \cos 2\pi f_0(2t+\tau)\right) + \frac{\sigma^2}{2}\left(\cos 2\pi f_0\tau - \cos 2\pi f_0(2t+\tau)\right) \tag{8}$$
$$= \sigma^2 \cos 2\pi f_0 \tau \tag{9}$$

Thus $R_W(t,\tau) = R_W(\tau)$. Since

$$E\left[W(t)\right] = E\left[X\right]\cos 2\pi f_0 t + E\left[Y\right]\sin 2\pi f_0 t = 0, \tag{10}$$

we can conclude that $W(t)$ is a wide sense stationary process. However, we note that if $E[X^2] \neq E[Y^2]$, then the $\cos 2\pi f_0(2t+\tau)$ terms in $R_W(t,\tau)$ would not cancel and $W(t)$ would not be wide sense stationary.

Problem 10.11.2 Solution

To show that $X(t)$ and $X_i(t)$ are jointly wide sense stationary, we must first show that $X_i(t)$ is wide sense stationary and then we must show that the cross correlation $R_{XX_i}(t,\tau)$ is only a function of the time difference τ. For each $X_i(t)$, we have to check whether these facts are implied by the fact that $X(t)$ is wide sense stationary.

(a) Since $E[X_1(t)] = E[X(t+a)] = \mu_X$ and

$$R_{X_1}(t,\tau) = E\left[X_1(t)X_1(t+\tau)\right] \tag{1}$$
$$= E\left[X(t+a)X(t+\tau+a)\right] \tag{2}$$
$$= R_X(\tau), \tag{3}$$

we have verified that $X_1(t)$ is wide sense stationary. Now we calculate the cross correlation

$$R_{XX_1}(t,\tau) = E\left[X(t)X_1(t+\tau)\right] \tag{4}$$
$$= E\left[X(t)X(t+\tau+a)\right] \tag{5}$$
$$= R_X(\tau+a). \tag{6}$$

Since $R_{XX_1}(t,\tau)$ depends on the time difference τ but not on the absolute time t, we conclude that $X(t)$ and $X_1(t)$ are jointly wide sense stationary.

(b) Since $E[X_2(t)] = E[X(at)] = \mu_X$ and

$$R_{X_2}(t,\tau) = E\left[X_2(t)X_2(t+\tau)\right] \tag{7}$$
$$= E\left[X(at)X(a(t+\tau))\right] \tag{8}$$
$$= E\left[X(at)X(at+a\tau)\right] = R_X(a\tau), \tag{9}$$

we have verified that $X_2(t)$ is wide sense stationary. Now we calculate the cross correlation

$$
\begin{aligned}
R_{XX_2}(t,\tau) &= E\left[X(t)X_2(t+\tau)\right] && (10)\\
&= E\left[X(t)X(a(t+\tau))\right] && (11)\\
&= R_X((a-1)t+\tau). && (12)
\end{aligned}
$$

Except for the trivial case when $a = 1$ and $X_2(t) = X(t)$, $R_{XX_2}(t,\tau)$ depends on both the absolute time t and the time difference τ, we conclude that $X(t)$ and $X_2(t)$ are not jointly wide sense stationary.

Problem 10.12.1 Solution

Writing $Y(t+\tau) = \int_0^{t+\tau} N(v)\,dv$ permits us to write the autocorrelation of $Y(t)$ as

$$
\begin{aligned}
R_Y(t,\tau) = E\left[Y(t)Y(t+\tau)\right] &= E\left[\int_0^t \int_0^{t+\tau} N(u)N(v)\,dv\,du\right] && (1)\\
&= \int_0^t \int_0^{t+\tau} E\left[N(u)N(v)\right]\,dv\,du && (2)\\
&= \int_0^t \int_0^{t+\tau} \alpha\delta(u-v)\,dv\,du. && (3)
\end{aligned}
$$

At this point, it matters whether $\tau \geq 0$ or if $\tau < 0$. When $\tau \geq 0$, then v ranges from 0 to $t+\tau$ and at some point in the integral over v we will have $v = u$. That is, when $\tau \geq 0$,

$$
R_Y(t,\tau) = \int_0^t \alpha\,du = \alpha t. \tag{4}
$$

When $\tau < 0$, then we must reverse the order of integration. In this case, when the inner integral is over u, we will have $u = v$ at some point so that

$$
R_Y(t,\tau) = \int_0^{t+\tau} \int_0^t \alpha\delta(u-v)\,du\,dv = \int_0^{t+\tau} \alpha\,dv = \alpha(t+\tau). \tag{5}
$$

Thus we see the autocorrelation of the output is

$$
R_Y(t,\tau) = \alpha \min\{t, t+\tau\} \tag{6}
$$

Perhaps surprisingly, $R_Y(t,\tau)$ is what we found in Example 10.19 to be the autocorrelation of a Brownian motion process. In fact, Brownian motion is the integral of the white noise process.