ECE 541
Stochastic Signals and Systems
Problem Set 1 Solutions
Sept 2005

Problem Solutions : Yates and Goodman, 1.4.4 1.4.5 1.4.7 1.5.6 1.6.5 1.6.7 1.7.7 1.8.7 and 1.9.4

Problem 1.4.4 Solution

Each statement is a consequence of part 4 of Theorem 1.4.

(a) Since $A \subset A \cup B$, $P[A] \leq P[A \cup B]$.

(b) Since $B \subset A \cup B$, $P[B] \leq P[A \cup B]$.

(c) Since $A \cap B \subset A$, $P[A \cap B] \leq P[A]$.

(d) Since $A \cap B \subset B$, $P[A \cap B] \leq P[B]$.

Problem 1.4.5 Solution

Specifically, we will use Theorem 1.7(c) which states that for any events A and B,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]. \tag{1}$$

To prove the union bound by induction, we first prove the theorem for the case of $n = 2$ events. In this case, by Theorem 1.7(c),

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]. \tag{2}$$

By the first axiom of probability, $P[A_1 \cap A_2] \geq 0$. Thus,

$$P[A_1 \cup A_2] \leq P[A_1] + P[A_2]. \tag{3}$$

which proves the union bound for the case $n = 2$. Now we make our induction hypothesis that the union-bound holds for any collection of $n - 1$ subsets. In this case, given subsets $A_1, \ldots, A_n$, we define

$$A = A_1 \cup A_2 \cup \cdots \cup A_{n-1}, \qquad B = A_n. \tag{4}$$

By our induction hypothesis,

$$P[A] = P[A_1 \cup A_2 \cup \cdots \cup A_{n-1}] \leq P[A_1] + \cdots + P[A_{n-1}]. \tag{5}$$

This permits us to write

$$\begin{aligned} P[A_1 \cup \cdots \cup A_n] &= P[A \cup B] && (6)\\ &\leq P[A] + P[B] \qquad \text{(by the union bound for } n = 2) && (7)\\ &= P[A_1 \cup \cdots \cup A_{n-1}] + P[A_n] && (8)\\ &\leq P[A_1] + \cdots P[A_{n-1}] + P[A_n] && (9) \end{aligned}$$

which completes the inductive proof.

Problem 1.4.7 Solution

It is tempting to use the following proof:

> Since S and ϕ are mutually exclusive, and since $S = S \cup \phi$,
>
> $$1 = P[S \cup \phi] = P[S] + P[\phi]. \tag{1}$$
>
> Since $P[S] = 1$, we must have $P[\phi] = 0$.

The above "proof" used the property that for mutually exclusive sets A_1 and A_2,

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]. \tag{2}$$

The problem is that this property is a consequence of the three axioms, and thus must be proven. For a proof that uses just the three axioms, let A_1 be an arbitrary set and for $n = 2, 3, \ldots$, let $A_n = \phi$. Since $A_1 = \cup_{i=1}^{\infty} A_i$, we can use Axiom 3 to write

$$P[A_1] = P[\cup_{i=1}^{\infty} A_i] = P[A_1] + P[A_2] + \sum_{i=3}^{\infty} P[A_i]. \tag{3}$$

By subtracting $P[A_1]$ from both sides, the fact that $A_2 = \phi$ permits us to write

$$P[\phi] + \sum_{n=3}^{\infty} P[A_i] = 0. \tag{4}$$

By Axiom 1, $P[A_i] \geq 0$ for all i. Thus, $\sum_{n=3}^{\infty} P[A_i] \geq 0$. This implies $P[\phi] \leq 0$. Since Axiom 1 requires $P[\phi] \geq 0$, we must have $P[\phi] = 0$.

Problem 1.5.6 Solution

The problem statement yields the obvious facts that $P[L] = 0.16$ and $P[H] = 0.10$. The words "10% of the ticks that had either Lyme disease or HGE carried both diseases" can be written as

$$P[LH|L \cup H] = 0.10. \tag{1}$$

(a) Since $LH \subset L \cup H$,

$$P[LH|L \cup H] = \frac{P[LH \cap (L \cup H)]}{P[L \cup H]} = \frac{P[LH]}{P[L \cup H]} = 0.10. \tag{2}$$

Thus,

$$P[LH] = 0.10 P[L \cup H] = 0.10 (P[L] + P[H] - P[LH]). \tag{3}$$

Since $P[L] = 0.16$ and $P[H] = 0.10$,

$$P[LH] = \frac{0.10 (0.16 + 0.10)}{1.1} = 0.0236. \tag{4}$$

(b) The conditional probability that a tick has HGE given that it has Lyme disease is

$$P[H|L] = \frac{P[LH]}{P[L]} = \frac{0.0236}{0.16} = 0.1475. \tag{5}$$

Problem 1.6.5 Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events

$$A_1 = \{1, 2\} \qquad A_2 = \{2, 3\} \qquad A_3 = \{3, 1\}. \tag{1}$$

Each event A_i has probability 1/2. Moreover, each pair of events is independent since

$$P[A_1A_2] = P[A_2A_3] = P[A_3A_1] = 1/4. \tag{2}$$

However, the three events A_1, A_2, A_3 are not independent since

$$P[A_1A_2A_3] = 0 \neq P[A_1]P[A_2]P[A_3]. \tag{3}$$

Problem 1.6.7 Solution

(a) For any events A and B, we can write the law of total probability in the form of

$$P[A] = P[AB] + P[AB^c]. \tag{1}$$

Since A and B are independent, $P[AB] = P[A]P[B]$. This implies

$$P[AB^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]. \tag{2}$$

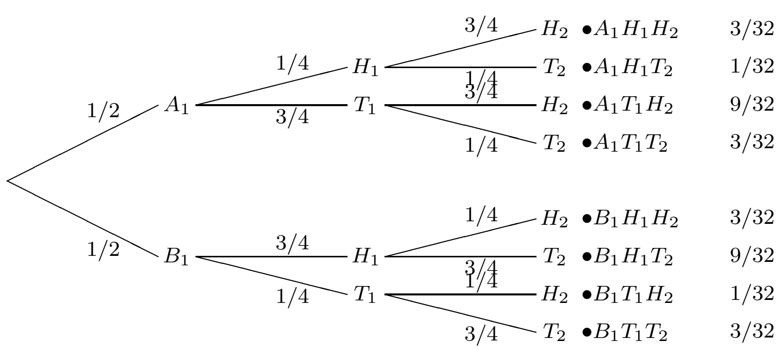
Thus A and B^c are independent.

(b) Proving that A^c and B are independent is not really necessary. Since A and B are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of A and B proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels A and B reversed.

(c) To prove that A^c and B^c are independent, we apply the result of part (a) to the sets A and B^c. Since we know from part (a) that A and B^c are independent, part (b) says that A^c and B^c are independent.

Problem 1.7.7 Solution

The tree for this experiment is



The event H_1H_2 that heads occurs on both flips has probability

$$P[H_1H_2] = P[A_1H_1H_2] + P[B_1H_1H_2] = 6/32. \tag{1}$$

The probability of H_1 is

$$P[H_1] = P[A_1H_1H_2] + P[A_1H_1T_2] + P[B_1H_1H_2] + P[B_1H_1T_2] = 1/2. \tag{2}$$

Similarly,

$$P[H_2] = P[A_1H_1H_2] + P[A_1T_1H_2] + P[B_1H_1H_2] + P[B_1T_1H_2] = 1/2. \tag{3}$$

Thus $P[H_1H_2] \neq P[H_1]P[H_2]$, implying H_1 and H_2 are not independent. This result should not be surprising since if the first flip is heads, it is likely that coin B was picked first. In this case, the second flip is less likely to be heads since it becomes more likely that the second coin flipped was coin A.

Problem 1.8.7 Solution

What our design must specify is the number of boxes on the ticket, and the number of specially marked boxes. Suppose each ticket has n boxes and $5+k$ specially marked boxes. Note that when $k > 0$, a winning ticket will still have k unscratched boxes with the special mark. A ticket is a winner if each time a box is scratched off, the box has the special mark. Assuming the boxes are scratched off randomly, the first box scratched off has the mark with probability $(5+k)/n$ since there are $5+k$ marked boxes out of n boxes. Moreover, if the first scratched box has the mark, then there are $4+k$ marked boxes out of $n-1$ remaining boxes. Continuing this argument, the probability that a ticket is a winner is

$$p = \frac{5+k}{n}\frac{4+k}{n-1}\frac{3+k}{n-2}\frac{2+k}{n-3}\frac{1+k}{n-4} = \frac{(k+5)!(n-5)!}{k!n!}. \tag{1}$$

By careful choice of n and k, we can choose p close to 0.01. For example,

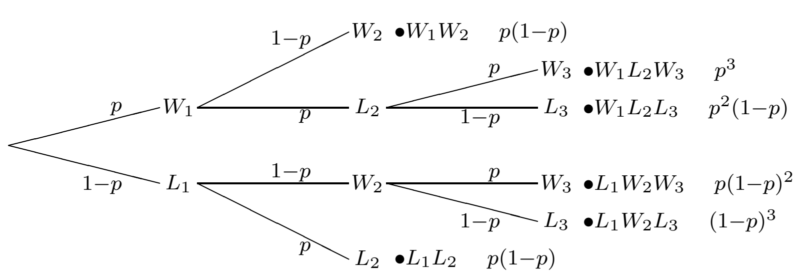
n	9	11	14	17
k	0	1	2	3
p	0.0079	0.012	0.0105	0.0090

(2)

A gamecard with $N = 14$ boxes and $5+k = 7$ shaded boxes would be quite reasonable.

Problem 1.9.4 Solution

For the team with the homecourt advantage, let W_i and L_i denote whether game i was a win or a loss. Because games 1 and 3 are home games and game 2 is an away game, the tree is



The probability that the team with the home court advantage wins is

$$P[H] = P[W_1 W_2] + P[W_1 L_2 W_3] + P[L_1 W_2 W_3] \tag{1}$$

$$= p(1-p) + p^3 + p(1-p)^2. \tag{2}$$

Note that $P[H] \leq p$ for $1/2 \leq p \leq 1$. Since the team with the home court advantage would win a 1 game playoff with probability p, the home court team is less likely to win a three game series than a 1 game playoff!