16:332:541 **Stochastic Signals and Systems** **Midterm Exam**

November 5, 2008

SOLUTION

You have 90 minutes to complete the first three problems of this exam. Find the limiting state probabilities. Items with unspecified point values are worth ten points. Please read both sides of the exam carefully and ask the instructor if you have any questions.

Preliminary: *(10 points)* Put your name and your Rutgers netid on the front of each exam bluebook. Write your random but memorable personal code (from the first quiz) on the *upper left corner of the inside front cover* of your first bluebook.

1. *40 points* Ten runners compete in a race starting at time $t = 0$. The runners' finishing times $R_1, \ldots, R_{10}$ are iid exponential random variables with expected value $1/\mu = 10$ minutes.

   (a) What is the probability that the last runner will finish in less than 20 minutes?
   The last runner's finishing time is $L = \max(R_1, \ldots, R_{10})$ *and*

   $$\begin{aligned} P[L \le 20] &= P[\max(R_1, \ldots, R_{10}) \le 20] \\ &= P[R_1 \le 20, R_2 \le 20, \ldots, R_{10} \le 20] \\ &= P[R_1 \le 20] P[R_2 \le 20] \cdots P[R_{10} \le 20] \\ &= (P[R_1 \le 20])^{10} \\ &= \left(1 - e^{-20\mu}\right)^{10} = \left(1 - e^{-2}\right)^{10} \approx 0.234 \end{aligned}$$

   (b) What is the PDF of X_1, the finishing time of the winning runner?
   At the start at time zero, we can view each runner as the first arrival of an independent Poisson process of rate μ. *Thus, at time zero, the arrival of the first runner can be viewed as the first arrival of a process of rate* 10μ. *Hence,* X_1 *is exponential with expected value* $1/(10\mu) = 1$ *and has PDF*

   $$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \ge 0, \\ 0 & \textit{otherwise.} \end{cases}$$

   (c) Find the PDF $f_Y(y)$ of $Y = R_1 + \cdots + R_{10}$.
   We can view Y *as the* $10th$ *arrival of a Poisson process of rate* μ. *Thus* Y *has the Erlang* $(n = 10, \mu)$ *PDF*

   $$f_Y(y) = \begin{cases} \frac{\mu^{10} y^9 e^{-\mu y}}{9!} & y \ge 0, \\ 0 & \textit{otherwise.} \end{cases}$$

   (d) Let $X_1, \ldots, X_{10}$ denote the runners' interarrival times at the finish line. Find the joint PDF $f_{X_1,\ldots,X_{10}}(x_1, \ldots, x_{10})$.
   We already found the PDF of X_1. *We observe that after the first runner finishes, there are still 9 runners on the course. Because each runner's time is memoryless, each runner has a residual running time that is an exponential* (μ) *random variable. Because these residual running times are independent,* X_2 *is exponential with expected value* $1/(9\mu) = 1/0.9$ *and has PDF*

   $$f_{X_2}(x_2) = \begin{cases} 9\mu e^{-9\mu x_2} & x_2 \ge 0, \\ 0 & \textit{otherwise,} \end{cases} = \begin{cases} 0.9 e^{-0.9 x_2} & x_2 \ge 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Similarly, for the ith arrival, there are $10-i+1=11-i$ *runners left on the course. The interarrival time for the ith arriving runner is the same as waiting for the first arrival ofa Poisson process of rate* $(11-i)\mu$. *Thus* X_i *has PDF*

$$f_{X_i}(x_i) = \begin{cases} (11-i)\mu e^{-(11-i)\mu x_i} & x_i \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Finally, we observe that the memeoryless property of the runners' exponential running times ensures that the X_i *are independent random variables. Hence,*

$$\begin{aligned} f_{X_1,\ldots,X_{10}}(x_1,\ldots,x_{10}) &= f_{X_1}(x_1) f_{X_2}(x_2)\cdots f_{X_{10}}(x_{10}) \\ &= \begin{cases} 10!\mu^{10} e^{-\mu(10x_1+9x_2+\cdots+2x_9+x_{10})} & x_i \geq 0, \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

2. *40 points* The Gaussian random vector $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ has expected value $E[\mathbf{X}] = \mathbf{0}$ and covariance matrix

$$\mathbf{C}_X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

(a) Find the PDF of $W = X_1 + 2X_2$.

Since $\mathbf{X}$ *is Gaussian,* W *is also Gaussian. THus we need only compute the expected value*

$$E[W] = E[X_1] + 2E[X_2] = 0$$

and variance

$$\begin{aligned} \sigma_W^2 = E\left[W^2\right] &= E\left[(X_1+2X_2)^2\right] \\ &= E\left[X_1^2 + 4X_1X_2 + 4X_2^2\right] \\ &= C_{11} + 4C_{12} + 4C_{22} = 10. \end{aligned}$$

Thus W *has the Gaussian* $(0, \sqrt{10})$ *PDF*

$$f_W(w) = \frac{1}{\sqrt{20\pi}} e^{-w^2/20}.$$

(b) Let $V = 2X_1$. Find the conditional PDF $f_{V|W}(v|w)$.

This is somewhat tricky to derive from scratch so this problem is mostly a reward for those with carefully constructed cheat sheets. In this case, we first calculate

$$\begin{aligned} E[V] &= 0, & \sigma_V^2 &= 4\sigma_{X_1}^2 = 8, \\ E[W] &= 0, & \sigma_W^2 &= 10, \end{aligned}$$

and that V *and* W *have correlation coefficient*

$$\rho_{VW} = \frac{E[VW]}{\sqrt{\sigma_V^2 \sigma_W^2}} = \frac{E[2X_1(X_1+2X_2)]}{\sqrt{80}} = \frac{2C_{11}+4C_{12}}{\sqrt{80}} = \frac{8}{\sqrt{80}} = \frac{2}{\sqrt{5}}.$$

Now we recall that the conditional PDF $f_{V|W}(v|w)$ *is Gaussian with conditional expected value*

$$E[V|W=w] = E[V] + \rho_{VW}\frac{\sigma_V}{\sigma_W}(w - E[W]) = \frac{2}{\sqrt{5}}\frac{\sqrt{8}}{\sqrt{10}} w = 4w/5$$

and conditional variance

$$\sigma^2_{V|W} = \sigma^2_V(1-\rho^2_{VW}) = \frac{8}{5}.$$

It follows that

$$f_{V|W}(v|w) = \frac{1}{\sqrt{2\pi\sigma^2_{V|W}}} e^{-(v-E[V|W])^2/2\sigma^2_{V|W}} = \sqrt{\frac{5}{16\pi}} e^{-5(v-4w/5)^2/16}.$$

(c) Find the PDF $f_{\mathbf{Y}}(\mathbf{y})$ of $\mathbf{Y} = \mathbf{A}\mathbf{X}$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$\mathbf{Y}$ *is a Gaussian random vector with* $E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] = \mathbf{0}$ *and covariance matrix*

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_X\mathbf{A}' = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}.$$

This implies $det(\mathbf{C}_{\mathbf{Y}}) = 4$ *and*

$$\mathbf{C}_{\mathbf{Y}}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}.$$

It follows that

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi[det(\mathbf{C}_{\mathbf{Y}})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{y}'\mathbf{C}_{\mathbf{Y}}^{-1}\mathbf{y}\right) = \frac{1}{4\pi} e^{-(y_1^2-2y_1y_2+5y_2^2)/8}.$$

(d) Does there exist a stationary Gaussian process $X(t)$ and time instances t_1 and t_2 such that $\mathbf{X}$ is actually a pair of observations $\begin{bmatrix} X(t_1) & X(t_2) \end{bmatrix}'$ of the process $X(t)$? Explain your answer.

The short answer is NO. If there were such a process $X(t)$*, then we would have*

$$\sigma^2_{X(t_1)} = C_{11} = 2, \qquad \sigma^2_{X(t_2)} = C_{22} = 1.$$

However, this is a contradiction since a stationarity of $X(t)$ *requires* $\sigma^2_{X(t_1)} = \sigma^2_{X(t_2)}$.

3. Packets arrive at a forwarding node as a Poisson process of rate 1 per millisecond (ms). The forwarder simply forwards (ie transmits) packets stored in its infinite capacity buffer. When the node is working, arriving packets are queued in the buffer and packet transmission times are independent exponential random variables with expected service time of $1/\mu = 0.5$ ms. However, the forwarding node takes a break after the completion of a packet transmission. This break has an exponential duration with expected value $1/\beta$ ms, independent of the arrival process and packet transmission times. During the break, the forwarder discards all arriving packets. Following the break, the node goes back to work by transmitting a previously buffered packet.

(a) Let M_1 denote the number of arriving packets during a one second interval of time. Find $E[M_1]$ and the PMF $P_{M_1}(m) = P[M_1 = m]$.

Since the arrival process is Poisson, the number of arrivals M_1 *in 1000 ms is a Poisson random variable with* $E[M] = 1000$ *and PMF*

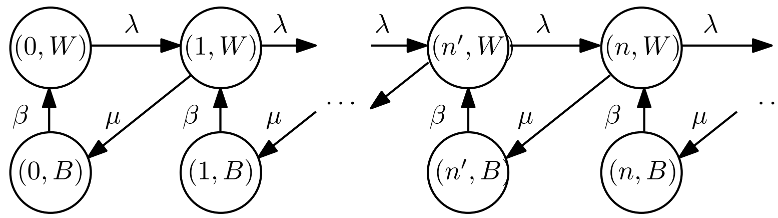
$$P_{M_1}(m) = \begin{cases} 1000^m e^{-1000}/m! & m = 0, 1, 2, \ldots \\ 0 & \textit{otherwise.} \end{cases}$$

(b) Sketch a continuous time Markov chain for this system. Hint: the forwarder may be *W*orking or on *B*reak when there are there are n buffered packets. For what values of β is the Markov chain irreducible?

The hint should make it clear that the states are

$$\{(0,W),(0,B),(1,W),(1,B),(2,W),(2,B)\ldots\}$$

where states (n,B) *and* (n,W) *correspond to* n *queued packets and indicating whether the server is on Break or Working. With* $\lambda=1$, $\mu=2$ *and using* $n'=n-1$, *the Markov chain is*



The Markov chain is irreducible as long as $\beta>0$. *When* $\beta=0$, *each state* (n,B) *belongs to its own communicating state because the process will get trapped in that state.*

(c) Find the limiting state probabilities when the chain is irreducible.

Partitioning the chain between $(n-1,W),(n-1,B)$ *and* (n,W), *we obtain* $p_{n-1,W}\lambda = p_{n,W}\mu$. *This implies*

$$p_{n,W}=\frac{\lambda}{\mu}p_{n-1,W} \qquad\Longrightarrow\qquad p_{n,W}=\rho^n p_{0,W}$$

where $\rho=\lambda/\mu$. *In addition, balancing rates in and out of state* $(n-1,B)$ *we obtain* $p_{n-1,B}\beta=p_{n,W}\mu$, *or*

$$p_{n-1,B}=\frac{\mu}{\beta}p_{n,W}=\frac{\lambda}{\beta}\rho^{n-1}p_{0,W}.$$

Equivalently, $p_{n,B}=(\lambda/\beta)\rho^n p_{0,W}$. *Normalizing so that the probabilities sum to* 1 *yields*

$$1=\sum_{n=0}^{\infty}(p_{n,W}+p_{n,B})=p_{0,W}\sum_{n=0}^{\infty}\left(\rho^n+\frac{\lambda}{\beta}\rho^n\right)=p_{0,W}\frac{\beta+\lambda}{\beta}\frac{1}{1-\rho}.$$

This implies

$$p_{0,W}=\frac{\beta(\mu-\lambda)}{\mu(\beta+\lambda)}. \tag{1}$$

Note that the chain is ergodic since $\lambda=1<\mu=2$.

(d) After the system has been running for a long time, what is the probability $P[D]$ that an arriving packet is discarded?

The event D *that an arrival is discarded occurs whenever the system is on break. After the system has been running a long time, this probability is given by the limiting state probability of being in a state* (n,B). *Hence,*

$$P[D]=\sum_{n=0}^{\infty}p_{n,B}=\frac{\lambda}{\beta}p_{0,W}\sum_{n=0}^{\infty}\rho^n=\frac{\lambda}{\beta+\lambda}.$$