16:332:541 Stochastic Signals and Systems Midterm Exam Solution

November 9, 2005

You have 100 minutes to complete the first three problems of this exam. You are invited to complete Problem 4 at home and to submit your solution in class on Monday. *The take home component must be completed alone without collaboration or assitance from other people.* Items with unspecified point values are worth ten points. Put your name and your Rutgers netid (but no part of your SSN) on each exam book *(10 points)*. Please read both sides of the exam carefully and ask the instructor if you have any questions.

1. *40 points* At time $t = 0$, the price of a stock is a constant k dollars. At time $t > 0$ the price of a stock is a Gaussian random variable X with $E[X] = k$ and $\sigma_X^2 = t$. At time t, a *Call Option at Strike* k has value

$$V = (X - k)^+$$

where the operator $(\cdot)^+$ is defined as $(z)^+ = \max(z, 0)$.

(a) *20 points* Find the moments $E[V]$ and $E[V^2]$.

Let $Y = X - k$. At time t, Y is a Gaussian $(0, \sqrt{t})$ random variable and since $V = Y^+$,

$$E[V] = E\left[Y^+\right] = \int_0^\infty y f_Y(y)\, dy = \frac{1}{\sqrt{2\pi t}} \int_0^\infty y e^{-y^2/2t}\, dy.$$

With the variable substitution $w = y^2/2t$, we have $dw = (y/t)\, dy$ and

$$E[V] = \frac{t}{\sqrt{2\pi t}} \int_0^\infty e^{-w}\, dw = \sqrt{\frac{t}{2\pi}}.$$

For the second moment of V,

$$E\left[V^2\right] = E\left[(Y^+)^2\right] = \int_0^\infty y^2 f_Y(y)\, dy.$$

However, $f_Y(y)$ is an even function satisfying $f_Y(y) = f_Y(-y)$, $g(y) = y^2 f_Y(y)$ is also an even function. This implies

$$E\left[V^2\right] = \frac{1}{2} \int_{-\infty}^\infty y^2 f_Y(y)\, dy = \frac{1}{2} E\left[Y^2\right] = \frac{1}{2}\sigma_Y^2 = \frac{t}{2}.$$

(b) Suppose you can buy the call option for d dollars at time $t = 0$. At time t, you can sell the call the call for V dollars and earn a profit (or loss perhaps) of $R = V - d$ dollars. Let d_0 denote the value of d such that

$P\{R > 0\} = 1/2$. Let d_1 denote the value of d such that $E[R] = 0$. Find d_0 and d_1.

$$P\{R > 0\} = P\{V - d > 0\} = P\{V > d\}.$$

Since V is nonnegative, $P\{V > d\} = 1$ for $d < 0$. Thus $d_0 \geq 0$ and for $d = d_0 \geq 0$,

$$P\{R_0 > 0\} = P\{(X-k)^+ > d_0\} = P\{X - k > d_0\} = P\left\{\frac{X-k}{\sqrt{t}} > \frac{d_0}{\sqrt{t}}\right\} = Q\left(\frac{d_0}{\sqrt{t}}\right).$$

Note that $Q(0) = 1/2$ and thus $d_0 = 0$. Finding d_1 is even simpler:

$$E[R] = E[V - d_1] = E[V] - d_1 = 0.$$

Thus,

$$d_1 = E[V] = \sqrt{\frac{t}{2\pi}}.$$

(c) Suppose $t = 30$ (days) and this experiment is repeated every month. At the start of a 30 day month, the stock price is k and you can buy a call option at strike k. However, since the price d of the call fluctuates every month, you decide to buy the call only if the price is no more than a threshold d^*. How should you choose your threshold d^*? Use probability theory to justify your answer.

Suppose the price d is a random variable D. Every month, we peroform the same experiment: we buy the option at the beginning of the month if $D \leq d^*$ and we sell at the end of the month for price V. In month n, our return will be a random variable R_n. After n months, our average return would be $(R_1 + \cdots + R_n)/n$. Since each month's experiment is independent, the R_n are an iid random sequence. By the weak law of large numbers, our time average monthly return $(R_1 + \cdots + R_n)/n$ will converge to $E[R]$. Note that this assumes that σ_R^2 is finite, however, this is straightforward to show. This justifies that we should choose the threshold d^* to maximize $E[R]$.

To proceed, we require a few additional assumptions. We will suppose that D and V are independent and that D has PDF $f_D(d)$. This assumption is not unreasonable because it assumes that the expected return $E[V]$ is embedded in the expected option price $E[D]$. Using τ to denote the threshold d^*, at the end of the thirty days, the return is

$$R = \begin{cases} V - D & D \leq \tau \\ 0 & D > \tau \end{cases}$$

Thus the conditional expected return is

$$E[R|D = x] = \begin{cases} E[V] - x & x \leq \tau \\ 0 & x > \tau \end{cases}$$

It follows that the expected return is

$$E[R] = \int_{-\infty}^{\infty} E[R|D = x] f_D(x)\, dx \qquad = \int_{-\infty}^{\tau} (E[V] - x)\, f_D(x)\, dx$$

To find the value of the threshold τ that maximizes $E[R]$, we calculate

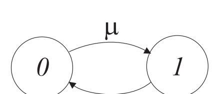
$$\frac{dE[R]}{d\tau} = (E[V] - \tau) f_D(\tau).$$

We see that $dE[R]/d\tau \geq 0$ for all $\tau \leq E[V]$ and that the derivative is zero at $\tau = E[V]$. Hence $E[R]$ is maximized at $\tau = d^* = E[V] = d_1$. In fact, this answer can be found by intuition. When the option has price $d < E[V]$, your choice is either to reject the option and earn 0 reward or to earn a reward R with $E[R] > 0$. On an expected value basis, it's always better to buy the call whenever $d < E[V]$. Hence we should set the threshold at $d^* = E[V]$.

2. *40 points* A remote sensor transmits measurement packets as a Poisson process of rate λ packets/sec to a data collection receiver through a random radio channel. The packets are very short so that we can asssume the sensor always completes the transmission of a packet well before a new packet is created. The radio channel alternates between good and bad states G and B. Good channel periods have an exponential duration with a mean value of $1/\gamma = 0.8$ sec while the duration of a bad period is exponential with mean $1/\mu = 0.4$ sec. The lengths of good and bad periods are all independent. At time 0, the system has been running for a long time and we start to count $R(t)$ the number of succesfully received packets in the interval $[0, t]$, Please complete the following parts:

   (a) Sketch a two state continuous time Markov chain for the radio channel. In steady state, what is the probability P_G that the channel is good at an arbitrary random time?

   Using 0 to denote the bad state B and 1 to denote the good state G, the two state Markov chain is



   The stationary probabilities must satisfy

$$p_0 \mu = p_1 \gamma.$$

   This implies $p_1 = (\mu/\gamma) p_0$. It follows from $p_0 + p_1 = 1$ that

$$p_0 = \frac{\gamma}{\gamma + \mu} = \frac{1}{3}, \qquad P_G = p_1 = \frac{\mu}{\gamma + \mu} = \frac{2}{3}.$$

(b) The sensor transmits measurement packets without examining the channel state. A packet will be received succesfully (without error) only if the entire transmission of the packet occurs during a period when the channel is good. Packets received in error are simply discarded by the receiver and no packets are ever retransmitted. If a packet has a deterministic transmission time t_0, what is the probability $P\{R_d\}$ that a measurement packet transmitted at a random time is received successfully?

A packet transmitted at time t is received successfully if the transmission starts in the good channel state and if the channel state stays in the good state for time duration t_0. Let $X(t)$ denote the channel state at time t. Given the channel is in the good state at time t, the time W until the channel makes a transition to the bad state is an exponential (γ) random variable. Assuming the channel state probabilities are given by the stationary probabilities at time t,

$$\begin{aligned} P\{R_d\} &= P\{X(t) = 1\}P\{W > t_0 | X(t) = 1\} \\ &= \frac{\mu}{\gamma + \mu} e^{-\gamma t_0} = \frac{2}{3} e^{-1.25 t_0} \end{aligned}$$

(c) Suppose now that the packet transmission time is an exponential random variable T with mean value of $1/\alpha$, now what is the probability $P\{R_e\}$ that a packet is received successfully?

Given that the channel is in the good state at time t, the packet is successfully transmitted if the packet completes transmission before the channel goes to the bad state. That is, the packet is successful if $X(t) = 1$ and $T < W$. This occurs with probability

$$P\{R_e\} = P\{X(t) = 1\}P\{T < W | X(t) = 1\}.$$

Given $X(t) = 1$, W, the time until the next state transition is an exponential (γ) random variable. Since T is exponential (α) and independent of W, we can view the event $T < W$ as an outcome of competing Poisson processes. An "arrival" of the packet arrival process occurs first if $T < W$; otherwise an "arrival" of the channel state transition process occurs first. This implies

$$P\{R_e\} = \frac{\mu}{\gamma + \mu} \frac{\alpha}{\alpha + \gamma} = \frac{2}{3} \frac{\alpha}{\alpha + \gamma}.$$

(d) Assuming again that the packet transmssion times are deterministic and short, is $R(t)$ a Poisson process? If so, justify your answer. If not, explain under what circumstances a Poisson model might be appropriate.

If the packet transmission time t_0 is fixed and short, we can assume the packet is always transmitted under a constant channel state. In this case, packets are always transmitted successfully in the good state and are never transmitted successfully in the bad channel state. To argue that the resulting packet delivery process is not Poisson, we have to argue that arrivals *of the process are not memoryless. Our approach is to show that a success at time t indicates the system is in the good state and will increase the likelihood of a success in the next interval. For a fixed but small value of Δ, let $R'(t) = R(t) - R(t - \Delta)$. Thus $R'(t) = 1$ indicates an arrival in the preceding interval of length Δ while $R'(t) = 0$ indicate no arrival in that same interval.*

$$\begin{aligned} P\{R'(t+\Delta) = 1 | R'(t) = 1\} &= \frac{P\{R'(t+\Delta) = 1, R'(t) = 1\}}{P\{R'(t) = 1\}} \\ &= \frac{p_1 \lambda\Delta(1 - \gamma\Delta)(\lambda\Delta)}{p_1\lambda\Delta} = \lambda\Delta. \end{aligned}$$

However,

$$\begin{aligned} P\{R'(t+\Delta) = 1 | R'(t) = 0\} &= \frac{P\{R'(t+\Delta) = 1, R'(t) = 0\}}{P\{R'(t) = 0\}} \\ &= \frac{p_0(\mu\Delta)(\lambda\Delta) + p_1\lambda\Delta(1-\lambda\Delta)}{p_1(1-\lambda\Delta) + p_0} \\ &= \frac{\lambda\Delta[p_0(\mu\Delta) + p_1(1-\lambda\Delta)]}{p_1(1-\lambda\Delta) + p_0} \end{aligned}$$

In the limit of small Δ, we neglect $O(\Delta)$ terms when added to $O(1)$ terms. This yields

$$\begin{aligned} P\{R'(t+\Delta) = 1 | R'(t) = 0\} = \frac{p_1\lambda\Delta}{p_0 + p_1} &= p_1\lambda\Delta \\ &= p_1 P\{R'(t+\Delta) = 1 | R'(t) = 1\} \end{aligned}$$

Thus, whether a packet was received in previous interval of size Δ changes the probability by the factor p_1 that a packet is received in the next size Δ interval. Note that when $\gamma = 99\mu$ the bad state has probability $p_0 = 0.99$ and the good state has probability $p_1 = 0.01$. In this case, the arrival in the previous interval increases the probability of an arrival in the next interval by a factor of 100. Hence the process is not Poisson.

However, if $t_0 \ll \lambda \ll \min(\gamma, \mu)$, then the channel state will change many times between packet arrivals. In this case, the channel state probabilities will settle to the stationary probabilities long before a packet arrives. Thus, each packet transmission will see the channel state chosen from the stationary distribution, independent of the past history. A transmitted packet will be successful with probability p_1, essentially independent of the past history of succesful transmissions. In this case, a Poisson model for $R(t)$ is quite reasonable.

3. *40 points* A wireless communication link transmits fixed-length packets. The transmission of a packet requires exactly one unit of time, called a "time slot." The wireless link is well designed so that a transmitted packet is always received correctly. We say the link is in the idle state in slot t if it has no packets to transmit in that slot. Here are some additional facts regarding the link:

(a) In each time slot t, a packet arrives with probability p, independent of the event of an arrival in any other slot and independent of the state of the system prior to its arrival.

(b) A packet arriving in slot t can be transmitted as early as slot $t+1$ if the link was busy in slot t. However, if the transmitter is idle in slot t, the new arriving packet must be queued while slots $t+1$ and $t+2$ are used for a link initialization procedure.

(c) Any additional packets that arrive during the initialization procedure are also queued until the initialization procedure is done.

Using 0 to denote the idle state, construct a discrete-time Markov chain for this system. Define (in words) what your system states represent. Calculate the stationary probability π_0 that the link is idle.

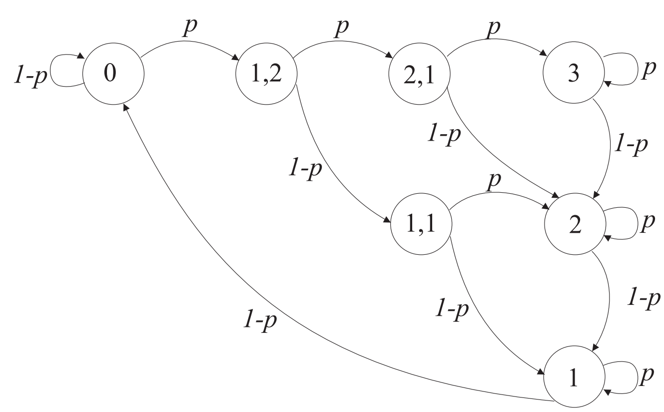
For the Markov chain, we use $n \in \{0,1,2,3\}$ to denote the state in which the system has n packets buffered and, if $n > 0$, is able to transmit a packet in the current slot. The system needs three additional states:

$1,2$ *The transmitter has* 1 *buffered packet but must wait* 2 *slots before transmitting.*

$1,1$ *The transmitter has* 1 *buffered packet but must wait* 1 *slot before transmitting.*

$2,1$ *The transmitter has* 2 *buffered packets but must wait* 1 *slot before transmitting.*

The resulting Markov chain is



The equations for the stationary probabilities are surprisingly simple. By omitting the equation for π_2 which has the most complicated set of incoming transi-

tions, we obtain

$$\begin{aligned}
\pi_0 &= (1-p)\pi_0 + (1-p)\pi_1 &\Rightarrow\quad& \pi_1 = \frac{p}{1-p}\pi_0 \\
\pi_{1,2} &= p\pi_0 && \\
\pi_{2,1} &= p\pi_{1,2} &\Rightarrow\quad& \pi_{2,1} = p^2\pi_0 \\
\pi_{1,1} &= (1-p)\pi_{1,2} &\Rightarrow\quad& \pi_{1,1} = p(1-p)\pi_0 \\
\pi_3 &= p\pi_{2,1} + p\pi_3 &\Rightarrow\quad& \pi_3 = \frac{p}{1-p}\pi_{2,1} = \frac{p^3}{1-p}\pi_0 \\
\pi_1 &= p\pi_1 + (1-p)\pi_{1,1} + (1-p)\pi_2 &\Rightarrow\quad& \pi_2 = \pi_1 - \pi_{1,1} = \frac{p^2(2-p)}{1-p}\pi_0
\end{aligned}$$

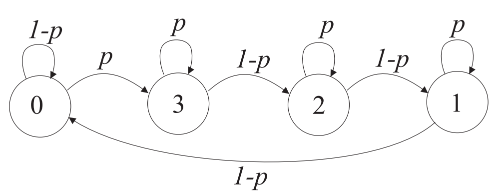
Applying the condition

$$\pi_0 + \pi_{1,2} + \pi_{2,1} + \pi_{1,1} + \pi_1 + \pi_2 + \pi_3 = 1$$

yields

$$\pi_0 = \frac{1-p}{1-2p}.$$

In fact, there is a reason the π_0 is so simple. Amar found a much simpler way to model the system state. The idea is that we can model the initialization period by requiring the system to transmit a dummy packet in each initialization slot. In this case, the state of the system is captured by the total number of buffered packets, including both real packets and dummy packets. The system is empty only when the ral packets and the dummy packets are sent. The simplified Markov chain is



The equations for this Markov chain are also much simpler.

$$\begin{aligned}
\pi_0 &= (1-p)\pi_0 + (1-p)\pi_1 &\Rightarrow\quad& \pi_1 = \frac{p}{1-p}\pi_0 \\
\pi_3 &= p\pi_0 + p\pi_3 &\Rightarrow\quad& \pi_3 = \frac{p}{1-p}\pi_0 \\
\pi_2 &= p\pi_2 + (1-p)\pi_3 &\Rightarrow\quad& \pi_2 = \pi_3 = \frac{p}{1-p}\pi_0
\end{aligned}$$

Applying the condition

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

yields $\pi_0 = (1-p)/(1-2p)$.

4. *80 points Take Home Problem* Random variables $X_1, X_2, \ldots$ are an iid random sequence. Each X_j has CDF $F_X(x)$ and PDF $f_X(x)$. Consider

$$L_n = \min(X_1, \ldots, X_n) \qquad\qquad U_n = \max(X_1, \ldots, X_n)$$

where labels L and U are chosen to remind of *L*ower and *U*pper. The following questions can be answered in terms of the CDF $F_X(x)$ and/or PDF $f_X(x)$.

(a) Find the CDF $F_{U_n}(u)$.
This is straightforward:

$$\begin{aligned} F_{U_n}(u) &= P\{\max(X_1, \ldots, X_n) \leq u\} \\ &= P\{X_1 \leq u, \ldots, X_n \leq u\} \\ &= P\{X_1 \leq u\}P\{X_2 \leq u\} \cdots P\{X_n \leq u\} = (F_X(u))^n \end{aligned}$$

(b) Find the CDF $F_{L_n}(l)$.
This is also straightforward.

$$\begin{aligned} F_{L_n}(l) &= 1 - P\{\min(X_1, \ldots, X_n) > l\} \\ &= 1 - P\{X_1 > l, \ldots, X_n > l\} \\ &= 1 - P\{X_1 > l\}P\{X_2 > l\} \cdots P\{X_n > l\} = 1 - (1 - F_X(l))^n \end{aligned}$$

(c) *20 points* Find the joint CDF $F_{L_n,U_n}(l, u)$.
This part is a little more difficult. The key is to identify the "easy" joint probability

$$\begin{aligned} P\{L_n > l, U_n \leq u\} &= P\{\min(X_1, \ldots, X_n) \geq l, \max(X_1, \ldots, X_n) \leq u\} \\ &= P\{l < X_i \leq u, i = 1, 2, \ldots, n\} \\ &= P\{l < X_1 \leq u\} \cdots P\{l < X_n \leq u\} \\ &= [F_X(u) - F_X(l)]^n \end{aligned}$$

Next we observe by the law of total probability that

$$P\{L_n > l, U_n \leq u\}P\{U_n \leq u\} = P\{L_n > l, U_n \leq u\} + P\{L_n \leq l, U_n \leq u\}.$$

The final term is the joint CDF we desire and using the expressions we derived for the first two terms, we obtain

$$\begin{aligned} F_{L_n,U_n}(l, u) &= P\{U_n \leq u\} - P\{L_n > l, U_n \leq u\} \\ &= [F_X(u)]^n - [F_X(u) - F_X(l)]^n \end{aligned}$$

(d) *40 points* Suppose the PDF $f_X(x)$ has the following properties

- $f_X(x) = f_X(-x)$
- $f_X(x) > 0$ for all x

A Gaussian $(0, \sigma)$ PDF would be one example (among many) of a PDF with the above properties. Suppose

$$R_n = \frac{U_n}{L_n}.$$

What properties can you deduce about R_n as n becomes large? You may wish to experiment with various PDFs in order to draw conclusions. It may well be that your conclusions will vary depending on the PDFs you examine. If you choose to work with specific PDFs, try to be clear what conclusions are generally true versus those conclusions that depend on the specific choice of PDF.

Comment: Parts (a)-(c) can be solved exactly. I don't actually know the answer(s) to part (d). You are welcome to look in the literature. Your grade will not be penalized. If you find papers or texts that are helpful in solving this problem, you must reference those works. I look forward to your investigations.

Several people observed experimentally that for large n that R_n has a fairly narrow PDF centered in the general vicinity of $r = -1$. However, nobody showed that the variance of R_n goes to zero, nor that the variance converges to some constant. Various ideas I had about analysis have not panned out so far. If I can make some progress on this, I'll try to revisit this on the final exam. :)