November 8, 2006

You have 90 minutes to complete the first three problems of this exam. You are invited to complete Problem 4 at home and to submit your solution in class on Monday. *The take home component must be completed alone without collaboration or assistance from other people.* Items with unspecified point values are worth ten points. Please read both sides of the exam carefully and ask the instructor if you have any questions.

Preliminary: *(10 points)* Put your name and your Rutgers netid on the front of each exam bluebook. Make up an ostensibly random (but personally memorable) four digit code. Write this code on the *upper right corner of the inside front cover* of your first bluebook. This code must not contain any part of your SSN or Rutgers ID number. (This code will be used to post grades at the end of the semester.

1. *40 points* Random variables X_1 and X_2 have zero expected value. The random vector $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ has a covariance matrix of the form

$$\mathbf{C} = \begin{bmatrix} 1 & \alpha \\ \beta & 4 \end{bmatrix}$$

   (a) For what values of α and β is $\mathbf{C}$ a valid covariance matrix?

   (b) For what values of α and β can $\mathbf{X}$ be a Gaussian random vector?

   (c) *20 points* Suppose now that α and β satisfy the conditions in part (b) and $\mathbf{X}$ is a Gaussian random vector.

      i. What is the PDF of X_2?

      ii. What is the PDF of $W = 2X_1 - X_2$?

2. *60 points* Starting infinitely long ago in the past, a new customer arrives each minute at a bank, exactly at the start of each minute. Each arriving customer is immediately served by a teller. (There are always as many tellers as needed to serve all customers in the bank.) After each minute of service, a customer departs with probability $1 - p$, independent of the departures of all other customers. Departures at the end of minute $t - 1$ occur the instant before the new customer arrives at the start of minute t. We say that a customer is in service at minute t, if the customer arrived at a minute $s \leq t$ and did not depart prior to the end of minute t.

   (a) Let $X_{t,k}(t)$ denote an indicator random variable such that $X_{t,k} = 1$ if the customer that arrived at the start of minute t is still in service at minute $t + k$; otherwise $X_{t,k} = 0$. What is the probability mass function (PMF) $P_{X_{t,k}}(x)$ of $X_{t,k}$?

   (b) *20 points* Let Y_t denote the number of customers in service at minute t (i.e. the instant after the new arrival at the start of minute t.) Find the expected value $E[Y_t]$ and variance $\sigma^2_{Y_t}$. Hint: $Y_t = X_{t,0} + X_{t-1,1} + \cdots$.

   (c) *30 points* Suppose that the customers are afraid of crowded places. Upon arrival, a customer instantly departs if the bank already has two customers. Model the process Y_t as a discrete-time Markov chain. Either find the limiting state probabilities or explain why they do not exist.

3. *70 points* At time $t = 0$, the price of a stock is a constant k dollars. At time $t > 0$, the price X of the stock is a uniform $(k - t, k + t)$ random variable. At time t, a *Put Option at Strike* k (which is the right to sell the stock at price k) has value $V = (k - X)^+$ where the operator $(\cdot)^+$ is defined as $(z)^+ = \max(z, 0)$. You may also recall that a *Call Option at Strike* k (the right to buy the stock at price k) has value $W = (X - k)^+$.

   (a) *20 points* At time 0, you sell the put and receive d dollars. At time t, you purchase the put for V dollars to cancel your position. Your profit is $R = d - V$. Find the central moments $E[R]$ and σ_R^2.

   (b) *20 points* In a short straddle, you sell the put for d dollars and you also sell the call for d dollars. At time t, you purchase the put for V dollars and the call for W dollars to cancel both positions. Your profit is

   $$R' = 2d - (V + W).$$

   Find the expected value $E[R']$ and variance $\sigma_{R'}^2$.

   (c) *20 points* Find the PDF $f_{R'}(r)$ of R'.

   (d) *10 points* Suppose d is sufficiently large that $E[R'] > 0$. Would you be interested in selling the short straddle? Are you getting something, namely $E[R']$ dollars, for nothing?

4. *70 points* Random variable $Y = X - Z$ is a noisy observation of the continuous random variable X. In particular, the noise Z has zero mean and unit variance and is independent of X. We wish to use Y to estimate X by forming an estimator of the form $\hat{X} = \hat{X}(Y) = aY$.

   (a) As a function of a and perhaps the moments of X, find the mean square error $e(a) = E\left[(X - \hat{X})^2\right]$.

   (b) Find $\hat{a}$, the value of a that minimizes the mean square estimation error $e(a)$.

   (c) Let b be an arbitrary constant. Show that the PDF of $W = b + Z$ is $f_W(w) = f_Z(w - b)$.

   (d) *20 points* Find the conditional expectation $E[X|Y]$.

   (e) *20 points* It is known that $X^*(Y) = E[X|Y]$ is the minimum mean square error estimator of X given Y. Consider the following argument:

   Since $X = Y + Z$, we see that if $Y = y$, then $X = y + Z$. Thus, from the result of part (c) with $W = X$ and $b = y$, the conditional PDF of X given $Y = y$ is $f_{X|Y}(x|y) = f_Z(x - y)$. It follows that

   $$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_{-\infty}^{\infty} x f_Z(x - y)\, dx.$$

   With the variable substitution, $z = x - y$, we obtain

   $$E[X|Y = y] = \int_{-\infty}^{\infty} (z + y) f_Z(z)\, dz = E[Z] + y = y.$$

   We conclude that $E[X|Y] = Y$. Since $E[X|Y]$ is optimal in the mean square square sense, we conclude that the optimal estimator of the form $\hat{X}(Y) = \hat{a}Y$ must satisfy $\hat{a} = 1$.

   Show that the above conclusion is inconsistent with your prior results. What is the error in the above argument?