16:332:541 **Stochastic Signals and Systems** **Midterm Exam**

November 9, 2005

You have 100 minutes to complete the first three problems of this exam. You are invited to complete Problem 4 at home and to submit your solution in class on Monday. *The take home component must be completed alone without collaboration or assitance from other people.* Items with unspecified point values are worth ten points. Put your name and your Rutgers netid (but no part of your SSN) on each exam book *(10 points)*. Please read both sides of the exam carefully and ask the instructor if you have any questions.

1. *40 points* At time $t = 0$, the price of a stock is a constant k dollars. At time $t > 0$ the price of a stock is a Gaussian random variable X with $E[X] = k$ and $\sigma_X^2 = t$. At time t, a *Call Option at Strike* k has value
$$V = (X - k)^+$$
where the operator $(\cdot)^+$ is defined as $(z)^+ = \max(z, 0)$.

   (a) *20 points* Find the moments $E[V]$ and $E\left[V^2\right]$.

   (b) Suppose you can buy the call option for d dollars at time $t = 0$. At time t, you can sell the call the call for V dollars and earn a profit (or loss perhaps) of $R = V - d$ dollars. Let d_0 denote the value of d such that $P\{R > 0\} = 1/2$. Let d_1 denote the value of d such that $E[R] = 0$. Find d_0 and d_1.

   (c) Suppose $t = 30$ (days) and this experiment is repeated every month. At the start of a 30 day month, the stock price is k and you can buy a call option at strike k. However, since the price d of the call fluctuates every month, you decide to buy the call only if the price is no more than a threshold d^*. How should you choose your threshold d^*? Use probability theory to justify your answer.

2. *40 points* A remote sensor transmits measurement packets as a Poisson process of rate λ packets/sec to a data collection receiver through a random radio channel. The packets are very short so that we can asssume the sensor always completes the transmission of a packet well before a new packet is created. The radio channel alternates between good and bad states G and B. Good channel periods have an exponential duration with a mean value of $1/\gamma = 0.8$ sec while the duration of a bad period is exponential with mean $1/\mu = 0.4$ sec. The lengths of good and bad periods are all independent. At time 0, the system has been running for a long time and we start to count $R(t)$ the number of succesfully received packets in the interval $[0, t]$, Please complete the following parts:

   (a) Sketch a two state continuous time Markov chain for the radio channel. In steady state, what is the probability P_G that the channel is good at an arbitrary random time?

   (b) The sensor transmits measurement packets without examining the channel state. A packet will be received succesfully (without error) only if the entire transmission of the packet occurs during a period when the channel is good. Packets received in error are simply discarded by the receiver and no packets are ever retransmitted. If a packet has a deterministic transmission time t_0, what is the probability $P\{R_d\}$ that a measurement packet transmitted at a random time is received successfully?

   (c) Suppose now that the packet transmission time is an exponential random variable T with mean value of $1/\alpha$, now what is the probability $P\{R_e\}$ that a packet is received successfully?

(d) Assuming again that the packet transmssion times are deterministic and short, is $R(t)$ a Poisson process? If so, justify your answer. If not, explain under what circumstances a Poisson model might be appropriate.

3. *40 points* A wireless communication link transmits fixed-length packets. The transmission of a packet requires exactly one unit of time, called a "time slot." The wireless link is well designed so that a transmitted packet is always received correctly. We say the link is in the idle state in slot t if it has no packets to transmit in that slot. Here are some additional facts regarding the link:

   (a) In each time slot t, a packet arrives with probability p, independent of the event of an arrival in any other slot and independent of the state of the system prior to its arrival.

   (b) A packet arriving in slot t can be transmitted as early as slot $t+1$ if the link was busy in slot t. However, if the transmitter is idle in slot t, the new arriving packet must be queued while slots $t+1$ and $t+2$ are used for a link initialization procedure.

   (c) Any additional packets that arrive during the initialization procedure are also queued until the initialization procedure is done.

   Using 0 to denote the idle state, construct a discrete-time Markov chain for this system. Define (in words) what your system states represent. Calculate the stationary probability π_0 that the link is idle.

4. *80 points* *Take Home Problem* Random variables $X_1, X_2, \dots$ are an iid random sequence. Each X_j has CDF $F_X(x)$ and PDF $f_X(x)$. Consider

$$L_n = \min(X_1, \dots, X_n) \qquad\qquad U_n = \max(X_1, \dots, X_n)$$

   where labels L and U are chosen to remind of *L*ower and *U*pper. The following questions can be answered in terms of the CDF $F_X(x)$ and/or PDF $f_X(x)$.

   (a) Find the CDF $F_{U_n}(u)$.

   (b) Find the CDF $F_{L_n}(l)$.

   (c) *20 points* Find the joint CDF $F_{L_n,U_n}(l, u)$.

   (d) *40 points* Suppose the PDF $f_X(x)$ has the following properties

   - $f_X(x) = f_X(-x)$
   - $f_X(x) > 0$ for all x

   A Gaussian $(0, \sigma)$ PDF would be one example (among many) of a PDF with the above properties. Suppose

$$R_n = \frac{U_n}{L_n}.$$

   What properties can you deduce about R_n as n becomes large? You may wish to experiment with various PDFs in order to draw conclusions. It may well be that your conclusions will vary depending on the PDFs you examine. If you choose to work with specific PDFs, try to be clear what conclusions are generally true versus those conclusions that depend on the specific choice of PDF.

   Comment: Parts (a)-(c) can be solved exactly. I don't actually know the answer(s) to part (d). You are welcome to look in the literature. Your grade will not be penalized. If you find papers or texts that are helpful in solving this problem, you must reference those works. I look forward to your investigations.