16:332:541 **Stochastic Signals and Systems** **Final Exam**

SOLUTION

December 22, 2006

You have 180 minutes to complete this exam. Put your name and your Rutgers netid (but no part of your SSN or Rutger ID) on each exam book *(10 points)*. Please read *both* sides of the exam carefully and answer **all four questions**. Ask the instructor if you have any questions.

x	0.0	0.5	1.0	1.5	2.0	2.5	3
$Q(x) = 1 - \Phi(x)$	0.500	0.3085	0.1587	0.0668	0.02275	0.00621	$1.35 \cdot 10^{-3}$

1. *30 points* TRUE OR FALSE. All answers must be justified. Keep in mind that for an answer to be TRUE, it must be true in all possible cases.

   (a) X_1 and X_2 are jointly Gaussian random variables. For any constant y, there exists a constant a such that $P[X_1 + aX_2 \le y] = 1/2$.
   FALSE: Let $W = X_1 + aX_2$. If $E[X_2] = 0$, then $E[W] = E[X_1]$ for all a. Since W is Gaussian, $P[W \le y] = 1/2$ if and only if $E[W] = E[X_1] = y$. We obtain a simple counterexample when $y = E[X_1] - 1$. Note that the answer would be true if we knew that $E[X_2]$ was nonzero. Also note that the variance of W will depend on the correlation between X_1 and X_2, but the correlation is irrelevant in the above argument.

   (b) For identically distributed zero-mean random variables Y_1 and Y_2, $\text{Var}[Y_1+Y_2] \ge \text{Var}[Y_1]$.
   FALSE: Suppose Y_1 and Y_2 have correlation coefficient $\rho = -3/4$. In this case,

   $$\text{Var}[Y_1 + Y_2] = \text{Var}[Y_1] + \text{Var}[Y_2] + 2\,\text{Cov}[Y_1, Y_2]. \tag{1}$$

   Since Y_1 and Y_2 are identically distributed, $\text{Var}[Y_1] = \text{Var}[Y_2] = \text{Var}[Y]$ *and* $\text{Cov}[Y_1, Y_2] = \rho\,\text{Var}[Y]$. *Thus,*

   $$\text{Var}[Y_1 + Y_2] = 2\,\text{Var}[Y] + 2\rho\,\text{Var}[Y] = \text{Var}[Y]/2 < \text{Var}[Y_1]. \tag{2}$$

   (c) If $X(t)$ and $Y(t)$ are independent zero-mean wide-sense stationary processes, then $W(t) = X(t) + Y(t)$ is wide-sense stationary.
   TRUE: First we observe that $E[W(t)] = E[X(t)] + E[Y(t)] = 0$. Next, we observe that

   $$\begin{aligned} R_W(t,\tau) &= E[W(t)W(t+\tau)] \\ &= E[(X(t)+Y(t))(X(t+\tau)+Y(t+\tau))] \\ &= E[X(t)X(t+\tau)] + E[Y(t)X(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)Y(t+\tau)]. \end{aligned}$$

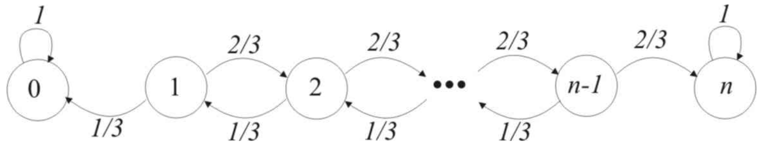
   Since $X(t)$ and $Y(t)$ are independent processes, $E[X(t_1)Y(t_2)] = E[X(t_1)]\,E[Y(t_2)] = 0$ for all t_1 and t_2. In the autocorrelation $R_W(t,\tau)$, the cross terms drop out and

   $$R_W(t,\tau) = R_X(\tau) + R_Y(\tau).$$

2. *60 points* Squares are labelled 1 through n consecutively from left to right. A player starts by placing a token on square k where $1 < k < n$. On each turn, a six-sided die is rolled. With a roll of 1 or 2, the token moves one square to the left. With a roll of 3 or higher, the token moves one square to the right. The player wins if the token reaches square 1. The game is a loss if the token reaches square n. When the game ends, the token stays on its final square (1 or n).

(a) Sketch a Markov chain that describes the position of the token and find the state transition matrix.

The Markov chain is



The state transition matrix is

$$
\mathbf{P} = \begin{bmatrix}
1 & 0 & 0 & & \cdots & & 0 \\
1/3 & 0 & 2/3 & 0 & & & \\
0 & 1/3 & 0 & 2/3 & 0 & & \vdots \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\vdots & & & 1/3 & 0 & 2/3 & 0 \\
 & & & 0 & 1/3 & 0 & 2/3 \\
0 & & \cdots & & 0 & 0 & 1
\end{bmatrix}
$$

(b) Identify any recurrent communicating classes, the set of transient states (if any).

Whenever you enter state 1, you remain there forever. Similarly, whenever you enter state n, you stay there. Thus $C_1 = \{1\}$ and $C_2 = \{n\}$ are recurrent communicating classes. All other states are transient. Note that $C_3 = \{2, 3, \ldots, n-1\}$ is a communicating class of transient states.

(c) Identify the set of all possible stationary distributions (if any exist).

There are stationary distributions These are

$$
\boldsymbol{\pi}_1 = \begin{bmatrix} 1 & 0 \cdots & 0 \end{bmatrix}', \qquad \boldsymbol{\pi}_2 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}'
$$

associated with the communicating classes C_1 and C_2. These stationary distributions correspond to getting locked into communicating class C_i forever. In terms of the game, the token reaches square 1 or square m and stays there.

Recall that a stationary distribution is defined as any probability vector such that $\boldsymbol{\pi}'\mathbf{P} = \boldsymbol{\pi}'$. You can verify that for any $0 \leq \alpha \leq 1$, $\boldsymbol{\pi} = \alpha\boldsymbol{\pi}_1 + (1-a)\boldsymbol{\pi}_2$ is also a stationary distribution.

(d) *20 points* Let W_k denote the event that you win the game starting from position k. Outline a calculation procedure to find $P[W_k]$ for a given value of k. You do not need to write matlab code, however you must provide a sequence of steps that a programmer could translate into matlab.

This problem can be viewed as being about Markov chain dynamics or about the limiting state probabilities in a system where the limiting state probabilities depend on the initial state probabilities. In this case we know that the state probability vector at step n is $\mathbf{p}(n)$ which satisfies $\mathbf{p}'(n) = \mathbf{p}'(0)\mathbf{P}^n$. where the initial state probability vector is $\mathbf{p}(0) = \mathbf{e}_k$, the Cartesian unit vector with a 1 is position k and zero elsewhere. Eventually, the token ends up either on square 1 or square n. This implies the limiting state probabilities are

$$
\boldsymbol{\pi}' = \lim_{n\to\infty} \mathbf{p}'(n) = \mathbf{e}_k' \lim_{n\to\infty} \mathbf{P}^n
$$

If we define $\mathbf{P}^\infty = \lim_{n\to\infty} \mathbf{P}^n$, then $\boldsymbol{\pi}'$ is the kth row of $\mathbf{P}^\infty$, a row vector of the form

$$
\boldsymbol{\pi}' = \begin{bmatrix} \omega & 0 & \cdots & 0 & 1-\omega \end{bmatrix}.
$$

where, if X_n denotes that token position at time n, $\omega = \lim_{n\to\infty} P[X_n = 1]$.

We can diagonalize $\mathbf{P}$ in the form $\mathbf{P}' = \mathbf{SDS}^{-1}$, where the columns of $\mathbf{S}$ are the right eigenvectors of $\mathbf{P}'$, or equivalently the transposed left eigenvectors of $\mathbf{P}$. Equivalently, we can write $\mathbf{P} = (\mathbf{S}')^{-1}\mathbf{DS}'$. The diagonal matrix $\mathbf{D} = diag[\lambda_1, \ldots, \lambda_n]$ will have eigenvalues $\lambda_1 = \lambda_2 = 1$, corresponding to the stationary distributions $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$. The other eigenvalues will have magnitude strictly less than 1. In this case

$$\mathbf{P}^n = (\mathbf{S}')^{-1}\mathbf{D}^n\mathbf{S}' \tag{3}$$

and $\mathbf{P}^\infty = (\mathbf{S}')^{-1}\mathbf{D}^\infty\mathbf{S}'$, where $\mathbf{D}^\infty = diag[1, 1, 0, \cdots, 0]$ because $\lim_{n\to\infty} \lambda_i^n = 0$ if $|\lambda_i| < 1$.

To find the limit in an algorithmic way, we would

- *Diagonalize $\mathbf{P}'$:* `[S,D]=eigs(P')`
- *Null out the non-unity entries in $\mathbf{D}$:* `DF=D.*(abs(D)>=1)`
- *Compute $\mathbf{P}^\infty$:* `PF=inv(S')*DF*(S')`

The $k, 1$ entry of `PF` *will hold the probability that you win starting from square k.*

(e) For the special case of $n = 4$, find $P[W_2]$ and $P[W_3]$. Hint: Don't use the calculation procedure from part (d).

The key observation is that anytime you are in position k, the probability that you eventually win is $P[W_k]$, independent of the past history of the game. By defining the events L and R for Left and Right moves of the token, we observe that

$$P[W_k|R] = P[W_{k-1}], \qquad P[W_k|L] = P[W_{k+1}].$$

Using the Law of total probability, we can write

$$\begin{aligned} P[W_k] &= P[W_k|L]\,P[L] + P[W_k|R]\,P[R] \\ &= P[W_{k-1}]\,(1/3) + P[W_{k+1}]\,(2/3) \end{aligned}$$

For the game with $n = 4$ squares, we know that $P[W_1] = 1$ and $P[W_4] = 0$. For squares 2 and 3, we can write

$$\begin{aligned} P[W_2] &= 1/3 + (2/3)P[W_3] \\ P[W_3] &= (1/3)P[W_2] \end{aligned}$$

We can solve these two equations for our two unknowns, yielding $P[W_2] = 3/7$ and $P[W_3] = 1/7$. This approach can be generalized to a set of equations of the form $\mathbf{w} = \mathbf{wQ}$ where $\mathbf{Q}$ is a matrix similar to $\mathbf{P}$. Also, for $n = 4$ states, the event W_2 occurs if and only if there are zero or more state 2 – state 3 – state 2 cycles followed by a transition to state 1. This leads to a sum that yields the above answers. Unfortunately, this method doesn't generalize for $n > 4$.

3. *50 points* The random sequence $X_1, X_2, \ldots$ is an iid sequence of Gaussian $(0, 1)$ random variables. $N(t)$ is a Poisson process of rate λ that is independent of the X_n. Let $\{Y(t)|t \geq 0\}$ denote a random process defined by $Y(t) = \sum_{n=0}^{N(t)} X_n$. Answers to the following questions must be justified.

(a) Find the conditional CDF $F_{Y(t)|N(t)}(y|n) = P[Y(t) \leq y|N(t) = n]$. Express your answer in terms of the $\Phi(\cdot)$ function.

Given $N(t) = n$, $Y(t) = X_0 + \cdots + X_n$ is a sum of $n+1$ Gaussian $(0, 1)$ random variables. Since

$$E[Y(t)] = (n+1)E[X] = 0, \qquad \mathrm{Var}[Y(t)] = (n+1)\,\mathrm{Var}[X] = n+1,$$

$Y(t)$ has conditional CDF

$$F_{Y(t)|N(t)}(y|n) = P[Y(t) \le y|N(t) = n] = P\left[\frac{Y(t)}{\sqrt{n+1}} \le \frac{y}{\sqrt{n+1}}|N(t) = n\right]$$
$$= \Phi\left(\frac{y}{\sqrt{n+1}}\right).$$

(b) Is $Y(t)$ a Gaussian process?

A necessary condition for $Y(t)$ to be a Gaussian process is that the random variable $Y(t)$ be Gaussian for every time instant t. In particular $Y(t)$ is Gaussian if it has a CDF of the form $F_{Y(t)}(y) = \Phi((y-\mu)/\sigma)$. For the given process, we can write the CDF of $Y(t)$ as

$$F_{Y(t)}(y) = P[Y(t) \le y] = \sum_{n=0}^{\infty} P[Y(t) \le y|N(t) = n]\, P[N(t) = n]$$
$$= \sum_{n=0}^{\infty} \Phi\left(\frac{y}{\sqrt{n+1}}\right) \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Unfortunately, this sum cannot be reduced to a single $\Phi(\cdot)$ function. If this is not clear, you should take a derivative of the CDF and see that you do not obtain a Gaussian PDF. Thus $Y(t)$ is not a Gaussian random variable and thus the process is not Gaussian.

(c) Is $Y(t)$ a stationary process?

The process is **not** *stationary because $F_{Y(t)}(y)$ depends on the time t.*

(d) *20 points* Is the process wide-sense stationary? Find the expected value function $\mu_Y(t) = E[Y(t)]$ and the autocovariance function $C_Y(t,\tau)$.

First we find the expected value and autocorrelation, and then we will know whether the process is wide-sense stationary. For the expected value, we recall from part (a) that conditioned on $N(t) = n$, $Y(t)$ was a zero mean Gaussian. That is, $E[Y(t)|N(t) = n] = 0$. This implies

$$E[Y(t)] = \sum_{n=0}^{\infty} E[Y(t)|N(t) = n]\, P[N(t) = n] = 0.$$

To find the autocovariance, we use the same trick as for finding the autocorrelation of the Poisson process and the Brownian motion process. To start, we assume $\tau > 0$. Since $E[Y(t)] = 0$,

$$C_Y(t,\tau) = E[Y(t)Y(t+\tau)] = E[Y(t)((Y(t+\tau) - Y(t)) + Y(t))]$$
$$= E[Y(t)(Y(t+\tau) - Y(t))] + E\left[Y^2(t)\right]. \qquad (4)$$

Note that $Y(t+\tau) - Y(t)$ depends on X_n corresponding to arrivals of the Poisson process in the interval $(t, t+\tau]$, which is independent of the arrivals prior to time t. In addition the X_n corresponding to arrivals in the interval $(t, t+\tau]$ are independent of the X_n corresponding to arrivals prior to time t. Thus $Y(t+\tau) - Y(t)$ is independent of $Y(t)$. It follows from (4) that

$$C_Y(t,\tau) = E[Y(t)]\, E[Y(t+\tau) - Y(t)] + E\left[Y^2(t)\right] = E\left[Y^2(t)\right] = \mathrm{Var}[Y(t)].$$

To calculate $\mathrm{Var}[Y(t)]$, we observe that it will be convenient to redefine $Y(t)$ as

$$Y(t) = X_1 + X_2 + \cdots + X_N,$$

where $N = N(t)+1$. This makes no difference since $Y(t)$ is still the sum of $N = N(t)+1$ iid Gaussian $(0,1)$ random variables. Written this way, we see that $Y(t)$ is a random sum of random variables such that N is independent of $X_1, X_2, \ldots$ The variance of a random sum of random variables is given by

$$\mathrm{Var}[Y(t)] = E[N]\,\mathrm{Var}[X] + \mathrm{Var}[N](E[X])^2.$$

Since $E[X] = 0$ and $\mathrm{Var}[X] = 1$, we have

$$\mathrm{Var}[Y(t)] = E[N] = E[N(t)+1] = \lambda t + 1.$$

This same result can also be obtained by careful use of the iterative expectation with conditioning on $N(t)$ and $N(t+\tau)$.
Thus for $\tau \geq 0$, $C_Y(t,\tau) = 1 + \lambda t$. For $\tau < 0$, $t+\tau < t$. In the above argument, we reverse all labels of t and $t+\tau$ and we can conclude that $C_Y(t,\tau) = 1 + \lambda(t+\tau)$. If you don't trust this argument, here are the details:

$$\begin{aligned} C_Y(t,\tau) = E[Y(t)Y(t+\tau)] &= E[(Y(t) - Y(t+\tau) + Y(t+\tau))Y(t+\tau)] \\ &= E[(Y(t) - Y(t+\tau))Y(t+\tau)] + E\left[Y^2(t+\tau)\right] \\ &= E\left[Y^2(t+\tau)\right] = \mathrm{Var}[Y(t+\tau)]. \end{aligned}$$

Note that $\mathrm{Var}[Y(t+\tau)]$ is the same as $\mathrm{Var}[Y(t)]$ but with t replace by $t+\tau$. Thus, for $\tau < 0$, $C_Y(t,\tau) = 1 + \lambda(t+\tau)$. A general expression for the autocovariance is

$$C_Y(t,\tau) = 1 + \lambda \min(t, t+\tau).$$

Since $C_Y(t,\tau)$ depends on t, $Y(t)$ is not wide-sense stationary.

4. *60 points* Suppose you have n suitcases. Suitcase i holds X_i dollars where $X_1, X_2, \ldots, X_n$ are iid continuous uniform $(0, m)$ random variables. (Think of a number like one **m**illion for the symbol m.) Unfortunately, you can't find out X_i until you open suitcase i.

   (a) Suppose you can open all n suitcases and then choose the suitcase with the most money. Let Y denote the amount you receive. What is $E[Y]$?
   If you can open all suitcases before choosing, $Y = \max(X_1, \ldots, X_n)$. To find $E[Y]$, we first find the CDF and PDF of Y:

$$\begin{aligned} F_Y(y) = P[\max(X_1,\ldots,X_n) \leq y] &= P[X_1 \leq y, \ldots, X_n \leq y] \\ &= (F_X(y))^n \\ &= \begin{cases} 0 & y < 0, \\ (y/m)^n & 0 \leq y \leq m, \\ 1 & m < y. \end{cases} \end{aligned}$$

   By taking the derivative of the CDF, the PDF of Y is

$$f_Y(y) = \begin{cases} ny^{n-1}/m^n & 0 \leq y \leq m, \\ 0 & \textit{otherwise.} \end{cases}$$

   The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \frac{n}{m^n} \int_0^m y^n\, dy = \frac{n}{(n+1)m^n} m^{n+1} = \frac{n}{n+1} m.$$

(b) Suppose you must open the suitcases one-by-one, starting with suitcase n and going down to suitcase 1. After opening suitcase i, you can either accept or reject X_i dollars. If you accept suitcase i, the game ends. If you reject, then you get to choose only from the still unopened suitcases.

What should you do? Perhaps it is not so obvious? In fact, you can decide before the game on a policy, a set of rules to follow. We will specify a policy by a vector $(\tau_1, \ldots, \tau_n)$ of threshold parameters.

- After opening suitcase i, you accept the amount X_i if $X_i \geq \tau_i$.
- Otherwise, you reject suitcase i and open suitcase $i-1$.
- If you have rejected suitcases n down through 2, then you must accept the amount X_1 in suitcase 1. Thus the threshold $\tau_1 = 0$ since you never reject the amount in the last suitcase.

i. Suppose you reject suitcases n through $i+1$, but then you accept suitcase i. Find $E[X_i|X_i \geq \tau_i]$.

You accept suitcase i when $X_i > \tau_i$. The values of past suitcases are irrelevant given tht you have opened suitcase i. Thus, given $X_i \geq \tau_i$, X_i has a conditional PDF of a continuous uniform (τ_i, m) random variable. Since a uniform (a, b) random variable has expected value $(a+b)/2$, we can conclude that

$$E[X_i|X_i \geq \tau_i] = \frac{\tau_i + m}{2}.$$

ii. Let W_k denote your reward given that there are k unopened suitcases remaining. What is $E[W_1]$?

When there is exactly one remaining suitcase, we must accept whatever reward it offers. Thus $W_1 = X_1$ and $E[W_1] = E[X_1] = m/2$.

iii. *20 points* As a function of τ_k, find a recursive relationship for $E[W_k]$ in terms of τ_k and $E[W_{k-1}]$.

In this case, we condition on whether $X_k \geq \tau_k$. Since $0 \leq \tau_k \leq m$, we can write

$$\begin{aligned} E[W_k] &= E[X_k|X_k \geq \tau_k]\, P[X_k \geq \tau_k] + E[W_{k-1}|X_k < \tau_k]\, P[X_k < \tau_k] \\ &= E[X_k|X_k \geq \tau_k]\, P[X_k \geq \tau_k] + E[W_{k-1}]\, P[X_k < \tau_k] \\ &= \frac{\tau_k + m}{2}\left(1 - \frac{\tau_k}{m}\right) + E[W_{k-1}]\,\frac{\tau_k}{m} \\ &= \frac{m^2 - \tau_k^2}{2m} + E[W_{k-1}]\,\frac{\tau_k}{m} \end{aligned} \tag{5}$$

iv. For $n = 4$ suitcases, find the optimal policy $(\tau_1^*, \ldots, \tau_4^*)$, that maximizes $E[W_4]$.

From the recursive relationship (5), we can find τ_k to maximize $E[W_k]$. In particular, solving

$$\frac{dE[W_k]}{d\tau_k} = -\frac{\tau_k}{m} + \frac{E[W_{k-1}]}{m} = 0$$

implies the optimal threshold is $\tau_k^* = E[W_{k-1}]$. That is, the optimal policy is to accept suitcase k if the reward X_k is higher than the expected reward we would receive from the remaining $k-1$ suitcases if we were to reject suitcase k. With one suitcase left, the optimal policy is $\tau_1^* = 0$ since we don't reject the last suitcase. The optimal reward is $E[W_1^*] = E[X_1] = m/2$. For two suitcases left, the optimal *threshold is* $\tau_2^* = E\left[W_1^*\right]$*. Using (5) we can recursively optimize the rewards:*

$$\begin{aligned} E\left[W_k^*\right] &= \frac{m^2 - (\tau_k^*)^2}{2m} + E\left[W_{k-1}^*\right]\frac{\tau_k^*}{m} \\ &= \frac{m}{2} - \frac{\left(E\left[W_{k-1}^*\right]\right)^2}{2m} + \frac{\left(E\left[W_{k-1}^*\right]\right)^2}{m} \\ &= \frac{m}{2} + \frac{\left(E\left[W_{k-1}^*\right]\right)^2}{2m} \end{aligned} \tag{6}$$

The recursion becomes more clear by defining $E\left[W_k^*\right] = m\alpha_k$*. Since the reward cannot exceed* m *dollars, we know that* $0 \le \alpha_k \le 1$*. In addition, it follows from (6) that*

$$\alpha_k = \frac{1}{2} + \frac{\alpha_{k-1}^2}{2}.$$

Since $\alpha_1 = 1/2$*,*

$$\begin{aligned} \alpha_2 &= \frac{1}{2} + \frac{1}{8} = \frac{5}{8}, \\ \alpha_3 &= \frac{1}{2} + \frac{\alpha_2^2}{2} = \frac{89}{128}, \\ \alpha_4 &= \frac{1}{2} + \frac{\alpha_3^2}{2} = \frac{24,305}{32768} = 0.74. \end{aligned}$$

The optimal thresholds are $\tau_1^* = 0$ *and for* $k > 1$*,* $\tau_k^* = E\left[W_{k-1}^*\right] = \alpha_{k-1}m$*. Thus,*

$$\tau_1^* = 0, \qquad \tau_2^* = \frac{5}{8}m \qquad \tau_3^* = \frac{89}{128}m, \qquad \tau_4^* = \frac{24305}{32768}m.$$

Note that if $\lim_{k\to\infty} \alpha_k = 1$*. That is, if the number of suitcases* k *goes to infinity, the optimal rewards and thresholds satisfy* $E\left[W_k^*\right] = \tau_{k-1}^* \to m$*.*