Stochastic Signals and Systems
Additional Solutions – Part 1

Problem Solutions : Yates and Goodman, 2.5.11 2.7.9 2.8.10 3.5.10 3.6.9 3.7.18 4.9.15 4.11.5 5.3.8 5.4.7 5.5.5 5.5.6 5.6.6 and 5.6.9

Problem 2.5.11 Solution

We write the sum as a double sum in the following way:

$$\sum_{i=0}^{\infty} P[X > i] = \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} P_X(j) \tag{1}$$

At this point, the key step is to reverse the order of summation. You may need to make a sketch of the feasible values for i and j to see how this reversal occurs. In this case,

$$\sum_{i=0}^{\infty} P[X > i] = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} P_X(j) = \sum_{j=1}^{\infty} j P_X(j) = E[X] \tag{2}$$

Problem 2.7.9 Solution

(a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9{,}366{,}819} \approx 1.07 \times 10^{-7} \tag{1}$$

(b) Assuming each ticket is chosen randomly, each of the $2n-1$ other tickets is independently a winner with probability q. The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1-q)^{2n-1-k} & k = 0, 1, \ldots, 2n-1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Since the pot has $n+r$ dollars, the expected amount that you win on your ticket is

$$E[V] = 0(1-q) + qE\left[\frac{n+r}{K_n+1}\right] = q(n+r)E\left[\frac{1}{K_n+1}\right] \tag{3}$$

Note that $E[1/K_n+1]$ was also evaluated in Problem 2.7.8. For completeness, we repeat those steps here.

$$E\left[\frac{1}{K_n+1}\right] = \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \tag{4}$$

$$= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)} \tag{5}$$

By factoring out $1/q$, we obtain

$$E\left[\frac{1}{K_n+1}\right] = \frac{1}{2nq}\sum_{k=0}^{2n-1}\binom{2n}{k+1}q^{k+1}(1-q)^{2n-(k+1)} \tag{6}$$

$$= \frac{1}{2nq}\underbrace{\sum_{j=1}^{2n}\binom{2n}{j}q^{j}(1-q)^{2n-j}}_{A} \tag{7}$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j=0$. Thus $A = 1-\binom{2n}{0}q^0(1-q)^{2n-0}$, which implies

$$E\left[\frac{1}{K_n+1}\right] = \frac{A}{2nq} = \frac{1-(1-q)^{2n}}{2nq} \tag{8}$$

The expected value of your ticket is

$$E[V] = \frac{q(n+r)[1-(1-q)^{2n}]}{2nq} = \frac{1}{2}\left(1+\frac{r}{n}\right)[1-(1-q)^{2n}] \tag{9}$$

Each ticket tends to be more valuable when the carryover pot r is large and the number of new tickets sold, $2n$, is small. For any fixed number n, corresponding to $2n$ tickets sold, a sufficiently large pot r will guarantee that $E[V] > 1$. For example if $n = 10^7$, (20 million tickets sold) then

$$E[V] = 0.44\left(1+\frac{r}{10^7}\right) \tag{10}$$

If the carryover pot r is 30 million dollars, then $E[V] = 1.76$. This suggests that buying a one dollar ticket is a good idea. This is an unusual situation because normally a carryover pot of 30 million dollars will result in far more than 20 million tickets being sold.

(c) So that we can use the results of the previous part, suppose there were $2n-1$ tickets sold before you must make your decision. If you buy one of each possible ticket, you are guaranteed to have one winning ticket. From the other $2n-1$ tickets, there will be K_n winners. The total number of winning tickets will be K_n+1. In the previous part we found that

$$E\left[\frac{1}{K_n+1}\right] = \frac{1-(1-q)^{2n}}{2nq} \tag{11}$$

Let R denote the expected return from buying one of each possible ticket. The pot had r dollars beforehand. The $2n-1$ other tickets are sold add $n-1/2$ dollars to the pot. Furthermore, you must buy $1/q$ tickets, adding $1/(2q)$ dollars to the pot. Since the cost of the tickets is $1/q$ dollars, your expected profit

$$E[R] = E\left[\frac{r+n-1/2+1/(2q)}{K_n+1}\right] - \frac{1}{q} \tag{12}$$

$$= \frac{q(2r+2n-1)+1}{2q} E\left[\frac{1}{K_n+1}\right] - \frac{1}{q} \tag{13}$$

$$= \frac{[q(2r+2n-1)+1](1-(1-q)^{2n})}{4nq^2} - \frac{1}{q} \tag{14}$$

For fixed n, sufficiently large r will make $E[R] > 0$. On the other hand, for fixed r, $\lim_{n\to\infty} E[R] = -1/(2q)$. That is, as n approaches infinity, your expected loss will be quite large.

Problem 2.8.10 Solution

We wish to minimize the function

$$e(\hat{x}) = E\left[(X-\hat{x})^2\right] \tag{1}$$

with respect to $\hat{x}$. We can expand the square and take the expectation while treating $\hat{x}$ as a constant. This yields

$$e(\hat{x}) = E\left[X^2 - 2\hat{x}X + \hat{x}^2\right] = E\left[X^2\right] - 2\hat{x}E[X] + \hat{x}^2 \tag{2}$$

Solving for the value of $\hat{x}$ that makes the derivative $de(\hat{x})/d\hat{x}$ equal to zero results in the value of $\hat{x}$ that minimizes $e(\hat{x})$. Note that when we take the derivative with respect to $\hat{x}$, both $E[X^2]$ and $E[X]$ are simply constants.

$$\frac{d}{d\hat{x}}\left(E\left[X^2\right] - 2\hat{x}E[X] + \hat{x}^2\right) = 2E[X] - 2\hat{x} = 0 \tag{3}$$

Hence we see that $\hat{x} = E[X]$. In the sense of mean squared error, the best guess for a random variable is the mean value. In Chapter 9 this idea is extended to develop minimum mean squared error estimation.

Problem 3.5.10 Solution

This problem is mostly calculus and only a little probability. From the problem statement, the SNR Y is an exponential $(1/\gamma)$ random variable with PDF

$$f_Y(y) = \begin{cases} (1/\gamma)e^{-y/\gamma} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, from the problem statement, the BER is

$$\overline{P}_e = E[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy = \int_0^{\infty} Q(\sqrt{2y}) \frac{y}{\gamma} e^{-y/\gamma}\, dy \tag{2}$$

Like most integrals with exponential factors, its a good idea to try integration by parts. Before doing so, we recall that if X is a Gaussian $(0,1)$ random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x). \tag{3}$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{dQ(x)}{dx} = -\frac{dF_X(x)}{dx} = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

To solve the integral, we use the integration by parts formula $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$, where

$$u = Q(\sqrt{2y}) \qquad dv = \frac{1}{\gamma} e^{-y/\gamma}\,dy \tag{5}$$

$$du = Q'(\sqrt{2y})\frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi y}} \qquad v = -e^{-y/\gamma} \tag{6}$$

From integration by parts, it follows that

$$\overline{P}_e = uv|_0^\infty - \int_0^\infty v\,du = -Q(\sqrt{2y})e^{-y/\gamma}\Big|_0^\infty - \int_0^\infty \frac{1}{\sqrt{y}} e^{-y[1+(1/\gamma)]}\,dy \tag{7}$$

$$= 0 + Q(0)e^{-0} - \frac{1}{2\sqrt{\pi}} \int_0^\infty y^{-1/2} e^{-y/\bar{\gamma}}\,dy \tag{8}$$

where $\bar{\gamma} = \gamma/(1+\gamma)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\bar{\gamma}$, we obtain

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \int_0^\infty t^{-1/2} e^{-t}\,dt \tag{9}$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}}\Gamma(1/2) = \frac{1}{2}\left[1 - \sqrt{\bar{\gamma}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma}{1+\gamma}}\right] \tag{10}$$

Problem 3.6.9 Solution

The professor is on time and lectures the full 80 minutes with probability 0.7. In terms of math,

$$P[T = 80] = 0.7. \tag{1}$$

Likewise when the professor is more than 5 minutes late, the students leave and a 0 minute lecture is observed. Since he is late 30% of the time and given that he is late, his arrival is uniformly distributed between 0 and 10 minutes, the probability that there is no lecture is

$$P[T = 0] = (0.3)(0.5) = 0.15 \tag{2}$$

The only other possible lecture durations are uniformly distributed between 75 and 80 minutes, because the students will not wait longer then 5 minutes, and that probability must add to a total of $1 - 0.7 - 0.15 = 0.15$. So the PDF of T can be written as

$$f_T(t) = \begin{cases} 0.15\delta(t) & t = 0 \\ 0.03 & 75 \le 7 < 80 \\ 0.7\delta(t-80) & t = 80 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 3.7.18 Solution

(a) Given $F_X(x)$ is a continuous function, there exists x_0 such that $F_X(x_0) = u$. For each value of u, the corresponding x_0 is unique. To see this, suppose there were also x_1 such that $F_X(x_1) = u$. Without loss of generality, we can assume $x_1 > x_0$ since otherwise we could exchange the points x_0 and x_1. Since $F_X(x_0) = F_X(x_1) = u$, the fact that $F_X(x)$ is nondecreasing implies $F_X(x) = u$ for all $x \in [x_0, x_1]$, i.e., $F_X(x)$ is flat over the interval $[x_0, x_1]$, which contradicts the assumption that $F_X(x)$ has no flat intervals. Thus, for any $u \in (0, 1)$, there is a unique x_0 such that $F_X(x) = u$. Moreiver, the same x_0 is the minimum of all x' such that $F_X(x') \geq u$. The uniqueness of x_0 such that $F_X(x)x_0 = u$ permits us to define $\tilde{F}(u) = x_0 = F_X^{-1}(u)$.

(b) In this part, we are given that $F_X(x)$ has a jump discontinuity at x_0. That is, there exists $u_0^- = F_X(x_0^-)$ and $u_0^+ = F_X(x_0^+)$ with $u_0^- < u_0^+$. Consider any u in the interval $[u_0^-, u_0^+]$. Since $F_X(x_0) = F_X(x_0^+)$ and $F_X(x)$ is nondecreasing,

$$F_X(x) \geq F_X(x_0) = u_0^+, \qquad x \geq x_0. \tag{1}$$

Moreover,

$$F_X(x) < F_X\left(x_0^-\right) = u_0^-, \qquad x < x_0. \tag{2}$$

Thus for any u satisfying $u_o^- \leq u \leq u_0^+$, $F_X(x) < u$ for $x < x_0$ and $F_X(x) \geq u$ for $x \geq x_0$. Thus, $\tilde{F}(u) = \min\{x | F_X(x) \geq u\} = x_0$.

(c) We note that the first two parts of this problem were just designed to show the properties of $\tilde{F}(u)$. First, we observe that

$$P\left[\hat{X} \leq x\right] = P\left[\tilde{F}(U) \leq x\right] = P\left[\min\left\{x' | F_X\left(x'\right) \geq U\right\} \leq x\right]. \tag{3}$$

To prove the claim, we define, for any x, the events

$$A: \quad \min\left\{x' | F_X\left(x'\right) \geq U\right\} \leq x, \tag{4}$$

$$B: \quad U \leq F_X(x). \tag{5}$$

Note that $P[A] = P[\hat{X} \leq x]$. In addition, $P[B] = P[U \leq F_X(x)] = F_X(x)$ since $P[U \leq u] = u$ for any $u \in [0, 1]$.

We will show that the events A and B are the same. This fact implies

$$P\left[\hat{X} \leq x\right] = P[A] = P[B] = P[U \leq F_X(x)] = F_X(x). \tag{6}$$

All that remains is to show A and B are the same. As always, we need to show that $A \subset B$ and that $B \subset A$.

- To show $A \subset B$, suppose A is true and $\min\{x' | F_X(x') \geq U\} \leq x$. This implies there exists $x_0 \leq x$ such that $F_X(x_0) \geq U$. Since $x_0 \leq x$, it follows from $F_X(x)$ being nondecreasing that $F_X(x_0) \leq F_X(x)$. We can thus conclude that

  $$U \leq F_X(x_0) \leq F_X(x). \tag{7}$$

  That is, event B is true.

- To show $B \subset A$, we suppose event B is true so that $U \leq F_X(x)$. We define the set
$$L = \left\{x' | F_X\left(x'\right) \geq U\right\}. \tag{8}$$
We note $x \in L$. It follows that the minimum element $\min\{x'|x' \in L\} \leq x$. That is,
$$\min\left\{x' | F_X\left(x'\right) \geq U\right\} \leq x, \tag{9}$$
which is simply event A.

Problem 4.9.15 Solution

If you construct a tree describing what type of call (if any) that arrived in any 1 millisecond period, it will be apparent that a fax call arrives with probability $\alpha = pqr$ or no fax arrives with probability $1-\alpha$. That is, whether a fax message arrives each millisecond is a Bernoulli trial with success probability α. Thus, the time required for the first success has the geometric PMF

$$P_T(t) = \begin{cases} (1-\alpha)^{t-1}\alpha & t = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that N is the number of trials required to observe 100 successes. Moreover, the number of trials needed to observe 100 successes is $N = T + N'$ where N' is the number of trials needed to observe successes 2 through 100. Since N' is just the number of trials needed to observe 99 successes, it has the Pascal $(k = 99, p)$ PMF

$$P_{N'}(n) = \binom{n-1}{98}\alpha^{99}(1-\alpha)^{n-99}. \tag{2}$$

Since the trials needed to generate successes 2 though 100 are independent of the trials that yield the first success, N' and T are independent. Hence

$$P_{N|T}(n|t) = P_{N'|T}(n-t|t) = P_{N'}(n-t). \tag{3}$$

Applying the PMF of N' found above, we have

$$P_{N|T}(n|t) = \binom{n-t-1}{98}\alpha^{99}(1-\alpha)^{n-t-99}. \tag{4}$$

Finally the joint PMF of N and T is

$$P_{N,T}(n,t) = P_{N|T}(n|t)\, P_T(t) \tag{5}$$
$$= \begin{cases} \binom{n-t-1}{98}\alpha^{100}(1-\alpha)^{n-100} & t = 1, 2, \ldots; n = 99+t, 100+t, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

This solution can also be found a consideration of the sample sequence of Bernoulli trials in which we either observe or do not observe a fax message.

To find the conditional PMF $P_{T|N}(t|n)$, we first must recognize that N is simply the number of trials needed to observe 100 successes and thus has the Pascal PMF

$$P_N(n) = \binom{n-1}{99}\alpha^{100}(1-\alpha)^{n-100} \tag{7}$$

Hence for any integer $n \geq 100$, the conditional PMF is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \frac{\binom{n-t-1}{98}}{\binom{n-1}{99}} & t = 1, 2, \ldots, n-99 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Problem 4.11.5 Solution

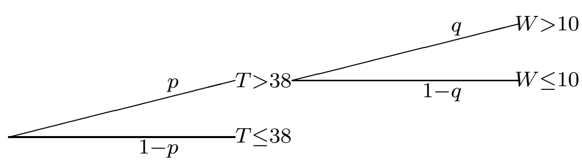
(a) The person's temperature is high with probability

$$p = P[T > 38] = P[T - 37 > 38 - 37] = 1 - \Phi(1) = 0.159. \tag{1}$$

Given that the temperature is high, then W is measured. Since $\rho = 0$, W and T are independent and

$$q = P[W > 10] = P\left[\frac{W-7}{2} > \frac{10-7}{2}\right] = 1 - \Phi(1.5) = 0.067. \tag{2}$$

The tree for this experiment is



The probability the person is ill is

$$P[I] = P[T > 38, W > 10] = P[T > 38]\,P[W > 10] = pq = 0.0107. \tag{3}$$

(b) The general form of the bivariate Gaussian PDF is

$$f_{W,T}(w,t) = \frac{\exp\left[-\frac{\left(\frac{w-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(w-\mu_1)(t-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{t-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \tag{4}$$

With $\mu_1 = E[W] = 7$, $\sigma_1 = \sigma_W = 2$, $\mu_2 = E[T] = 37$ and $\sigma_2 = \sigma_T = 1$ and $\rho = 1/\sqrt{2}$, we have

$$f_{W,T}(w,t) = \frac{1}{2\pi\sqrt{2}} \exp\left[-\frac{(w-7)^2}{4} - \frac{\sqrt{2}(w-7)(t-37)}{2} + (t-37)^2\right] \tag{5}$$

To find the conditional probability $P[I|T=t]$, we need to find the conditional PDF of W given $T=t$. The direct way is simply to use algebra to find

$$f_{W|T}(w|t) = \frac{f_{W,T}(w,t)}{f_T(t)} \tag{6}$$

The required algebra is essentially the same as that needed to prove Theorem 4.29. Its easier just to apply Theorem 4.29 which says that given $T=t$, the conditional distribution of W is Gaussian with

$$E[W|T=t] = E[W] + \rho\frac{\sigma_W}{\sigma_T}(t - E[T]) \tag{7}$$

$$\text{Var}[W|T=t] = \sigma_W^2(1-\rho^2) \tag{8}$$

Plugging in the various parameters gives

$$E[W|T=t] = 7 + \sqrt{2}(t-37) \quad \text{and} \quad \text{Var}[W|T=t] = 2 \tag{9}$$

Using this conditional mean and variance, we obtain the conditional Gaussian PDF

$$f_{W|T}(w|t) = \frac{1}{\sqrt{4\pi}} e^{-\left(w-(7+\sqrt{2}(t-37))\right)^2/4}. \tag{10}$$

Given $T=t$, the conditional probability the person is declared ill is

$$P[I|T=t] = P[W > 10|T=t] \tag{11}$$

$$= P\left[\frac{W-(7+\sqrt{2}(t-37))}{\sqrt{2}} > \frac{10-(7+\sqrt{2}(t-37))}{\sqrt{2}}\right] \tag{12}$$

$$= P\left[Z > \frac{3-\sqrt{2}(t-37)}{\sqrt{2}}\right] = Q\left(\frac{3\sqrt{2}}{2} - (t-37)\right). \tag{13}$$

Problem 5.3.8 Solution

In Problem 5.3.2, we found that the joint PMF of $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}'$ is

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3} & k_1 < k_2 < k_3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In this problem, we generalize the result to n messages.

(a) For $k_1 < k_2 < \cdots < k_n$, the joint event

$$\{K_1 = k_1, K_2 = k_2, \cdots, K_n = k_n\} \tag{2}$$

occurs if and only if all of the following events occur

A_1 $k_1 - 1$ failures, followed by a successful transmission
A_2 $(k_2 - 1) - k_1$ failures followed by a successful transmission
A_3 $(k_3 - 1) - k_2$ failures followed by a successful transmission
$\vdots$
A_n $(k_n - 1) - k_{n-1}$ failures followed by a successful transmission

Note that the events $A_1, A_2, \ldots, A_n$ are independent and

$$P[A_j] = (1-p)^{k_j - k_{j-1} - 1} p. \tag{3}$$

Thus

$$\begin{aligned} P_{K_1,\ldots,K_n}(k_1,\ldots,k_n) &= P[A_1]\,P[A_2]\cdots P[A_n] & (4)\\ &= p^n (1-p)^{(k_1-1)+(k_2-k_1-1)+(k_3-k_2-1)+\cdots+(k_n-k_{n-1}-1)} & (5)\\ &= p^n(1-p)^{k_n-n} & (6)\end{aligned}$$

To clarify subsequent results, it is better to rename $\mathbf{K}$ as $\mathbf{K}_n = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}'$. We see that

$$P_{\mathbf{K}_n}(\mathbf{k}_n) = \begin{cases} p^n(1-p)^{k_n-n} & 1 \le k_1 < k_2 < \cdots < k_n, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(b) For $j < n$,

$$P_{K_1,K_2,\ldots,K_j}(k_1,k_2,\ldots,k_j) = P_{\mathbf{K}_j}(\mathbf{k}_j). \tag{8}$$

Since $\mathbf{K}_j$ is just $\mathbf{K}_n$ with $n = j$, we have

$$P_{\mathbf{K}_j}(\mathbf{k}_j) = \begin{cases} p^j(1-p)^{k_j-j} & 1 \le k_1 < k_2 < \cdots < k_j, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

(c) Rather than try to deduce $P_{K_i}(k_i)$ from the joint PMF $P_{\mathbf{K}_n}(\mathbf{k}_n)$, it is simpler to return to first principles. In particular, K_i is the number of trials up to and including the ith success and has the Pascal (i, p) PMF

$$P_{K_i}(k_i) = \binom{k_i - 1}{i - 1} p^i (1-p)^{k_i - i}. \tag{10}$$

Problem 5.4.7 Solution

Since $U_1, \ldots, U_n$ are iid uniform $(0,1)$ random variables,

$$f_{U_1,\ldots,U_n}(u_1,\ldots,u_n) = \begin{cases} 1/T^n & 0 \le u_i \le 1; i = 1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since $U_1, \ldots, U_n$ are continuous, $P[U_i = U_j] = 0$ for all $i \neq j$. For the same reason, $P[X_i = X_j] = 0$ for $i \neq j$. Thus we need only to consider the case when $x_1 < x_2 < \cdots < x_n$.

To understand the claim, it is instructive to start with the $n = 2$ case. In this case, $(X_1, X_2) = (x_1, x_2)$ (with $x_1 < x_2$) if either $(U_1, U_2) = (x_1, x_2)$ or $(U_1, U_2) = (x_2, x_1)$. For infinitesimal Δ,

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2)\,\Delta^2 &= P[x_1 < X_1 \le x_1 + \Delta, x_2 < X_2 \le x_2 + \Delta] & (2)\\ &= P[x_1 < U_1 \le x_1 + \Delta, x_2 < U_2 \le x_2 + \Delta] \\ &\quad + P[x_2 < U_1 \le x_2 + \Delta, x_1 < U_2 \le x_1 + \Delta] & (3)\\ &= f_{U_1,U_2}(x_1,x_2)\,\Delta^2 + f_{U_1,U_2}(x_2,x_1)\,\Delta^2 & (4)\end{aligned}$$

We see that for $0 \le x_1 < x_2 \le 1$ that

$$f_{X_1,X_2}(x_1,x_2) = 2/T^n. \tag{5}$$

For the general case of n uniform random variables, we define $\boldsymbol{\pi} = \begin{bmatrix}\pi(1) & \ldots & \pi(n)\end{bmatrix}'$ as a permutation vector of the integers $1, 2, \ldots, n$ and Π as the set of $n!$ possible permutation vectors. In this case, the event $\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\}$ occurs if

$$U_1 = x_{\pi(1)}, U_2 = x_{\pi(2)}, \ldots, U_n = x_{\pi(n)} \tag{6}$$

for any permutation $\boldsymbol{\pi} \in \Pi$. Thus, for $0 \le x_1 < x_2 < \cdots < x_n \le 1$,

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\,\Delta^n = \sum_{\boldsymbol{\pi}\in\Pi} f_{U_1,\ldots,U_n}\left(x_{\pi(1)},\ldots,x_{\pi(n)}\right)\Delta^n. \tag{7}$$

Since there are $n!$ permutations and $f_{U_1,\ldots,U_n}(x_{\pi(1)},\ldots,x_{\pi(n)}) = 1/T^n$ for each permutation $\boldsymbol{\pi}$, we can conclude that

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = n!/T^n. \tag{8}$$

Since the order statistics are necessarily ordered, $f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = 0$ unless $x_1 < \cdots < x_n$.

Problem 5.5.5 Solution

Since 50 cents of each dollar ticket is added to the jackpot,

$$J_{i-1} = J_i + \frac{N_i}{2} \tag{1}$$

Given $J_i = j$, N_i has a Poisson distribution with mean j. It follows that $E[N_i|J_i = j] = j$ and that $\text{Var}[N_i|J_i = j] = j$. This implies

$$E\left[N_i^2|J_i = j\right] = \text{Var}[N_i|J_i = j] + \left(E\left[N_i|J_i = j\right]\right)^2 = j + j^2 \tag{2}$$

In terms of the conditional expectations given J_i, these facts can be written as

$$E\left[N_i|J_i\right] = J_i \qquad E\left[N_i^2|J_i\right] = J_i + J_i^2 \tag{3}$$

This permits us to evaluate the moments of J_{i-1} in terms of the moments of J_i. Specifically,

$$E\left[J_{i-1}|J_i\right] = E\left[J_i|J_i\right] + \frac{1}{2}E\left[N_i|J_i\right] = J_i + \frac{J_i}{2} = \frac{3J_i}{2} \tag{4}$$

This implies

$$E\left[J_{i-1}\right] = \frac{3}{2}E\left[J_i\right] \tag{5}$$

We can use this the calculate $E[J_i]$ for all i. Since the jackpot starts at 1 million dollars, $J_6 = 10^6$ and $E[J_6] = 10^6$. This implies

$$E\left[J_i\right] = (3/2)^{6-i}10^6 \tag{6}$$

Now we will find the second moment $E[J_i^2]$. Since $J_{i-1}^2 = J_i^2 + N_i J_i + N_i^2/4$, we have

$$\begin{aligned} E\left[J_{i-1}^2|J_i\right] &= E\left[J_i^2|J_i\right] + E\left[N_i J_i|J_i\right] + E\left[N_i^2|J_i\right]/4 && (7)\\ &= J_i^2 + J_i E\left[N_i|J_i\right] + (J_i + J_i^2)/4 && (8)\\ &= (3/2)^2 J_i^2 + J_i/4 && (9)\end{aligned}$$

By taking the expectation over J_i we have

$$E\left[J_{i-1}^2\right] = (3/2)^2 E\left[J_i^2\right] + E\left[J_i\right]/4 \tag{10}$$

This recursion allows us to calculate $E[J_i^2]$ for $i = 6, 5, \ldots, 0$. Since $J_6 = 10^6$, $E[J_6^2] = 10^{12}$. From the recursion, we obtain

$$E\left[J_5^2\right] = (3/2)^2 E\left[J_6^2\right] + E\left[J_6\right]/4 = (3/2)^2 10^{12} + \frac{1}{4}10^6 \tag{11}$$

$$E\left[J_4^2\right] = (3/2)^2 E\left[J_5^2\right] + E\left[J_5\right]/4 = (3/2)^4 10^{12} + \frac{1}{4}\left[(3/2)^2 + (3/2)\right]10^6 \tag{12}$$

$$E\left[J_3^2\right] = (3/2)^2 E\left[J_4^2\right] + E\left[J_4\right]/4 = (3/2)^6 10^{12} + \frac{1}{4}\left[(3/2)^4 + (3/2)^3 + (3/2)^2\right]10^6 \tag{13}$$

The same recursion will also allow us to show that

$$E\left[J_2^2\right] = (3/2)^8 10^{12} + \frac{1}{4}\left[(3/2)^6 + (3/2)^5 + (3/2)^4 + (3/2)^3\right]10^6 \tag{14}$$

$$E\left[J_1^2\right] = (3/2)^{10} 10^{12} + \frac{1}{4}\left[(3/2)^8 + (3/2)^7 + (3/2)^6 + (3/2)^5 + (3/2)^4\right]10^6 \tag{15}$$

$$E\left[J_0^2\right] = (3/2)^{12} 10^{12} + \frac{1}{4}\left[(3/2)^{10} + (3/2)^9 + \cdots + (3/2)^5\right]10^6 \tag{16}$$

Finally, day 0 is the same as any other day in that $J = J_0 + N_0/2$ where N_0 is a Poisson random variable with mean J_0. By the same argument that we used to develop recursions for $E[J_i]$ and $E[J_i^2]$, we can show

$$E\left[J\right] = (3/2)E\left[J_0\right] = (3/2)^7 10^6 \approx 17 \times 10^6 \tag{17}$$

and

$$\begin{aligned} E\left[J^2\right] &= (3/2)^2 E\left[J_0^2\right] + E\left[J_0\right]/4 && (18)\\ &= (3/2)^{14} 10^{12} + \frac{1}{4}\left[(3/2)^{12} + (3/2)^{11} + \cdots + (3/2)^6\right]10^6 && (19)\\ &= (3/2)^{14} 10^{12} + \frac{10^6}{2}(3/2)^6[(3/2)^7 - 1] && (20)\end{aligned}$$

Finally, the variance of J is

$$\mathrm{Var}[J] = E\left[J^2\right] - (E\left[J\right])^2 = \frac{10^6}{2}(3/2)^6[(3/2)^7 - 1] \tag{21}$$

Since the variance is hard to interpret, we note that the standard deviation of J is $\sigma_J \approx 9572$. Although the expected jackpot grows rapidly, the standard deviation of the jackpot is fairly small.

Problem 5.5.6 Solution

Let A denote the event $X_n = \max(X_1, \ldots, X_n)$. We can find $P[A]$ by conditioning on the value of X_n.

$$\begin{aligned}
P[A] &= P[X_1 \leq X_n, X_2 \leq X_n, \cdots, X_{n_1} \leq X_n] && (1)\\
&= \int_{-\infty}^{\infty} P[X_1 < X_n, X_2 < X_n, \cdots, X_{n-1} < X_n | X_n = x] f_{X_n}(x)\, dx && (2)\\
&= \int_{-\infty}^{\infty} P[X_1 < x, X_2 < x, \cdots, X_{n-1} < x | X_n = x] f_X(x)\, dx && (3)
\end{aligned}$$

Since $X_1, \ldots, X_{n-1}$ are independent of X_n,

$$P[A] = \int_{-\infty}^{\infty} P[X_1 < x, X_2 < x, \cdots, X_{n-1} < x] f_X(x)\, dx. \tag{4}$$

Since $X_1, \ldots, X_{n-1}$ are iid,

$$\begin{aligned}
P[A] &= \int_{-\infty}^{\infty} P[X_1 \leq x] P[X_2 \leq x] \cdots P[X_{n-1} \leq x] f_X(x)\, dx && (5)\\
&= \int_{-\infty}^{\infty} [F_X(x)]^{n-1} f_X(x)\, dx = \frac{1}{n} [F_X(x)]^n \Big|_{-\infty}^{\infty} = \frac{1}{n}(1-0) && (6)
\end{aligned}$$

Not surprisingly, since the X_i are identical, symmetry would suggest that X_n is as likely as any of the other X_i to be the largest. Hence $P[A] = 1/n$ should not be surprising.

Problem 5.6.6 Solution

This problem is quite difficult unless one uses the observation that the vector $\mathbf{K}$ can be expressed in terms of the vector $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ where J_i is the number of transmissions of message i. Note that we can write

$$\mathbf{K} = \mathbf{A}\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{J} \tag{1}$$

We also observe that since each transmission is an independent Bernoulli trial with success probability p, the components of $\mathbf{J}$ are iid geometric (p) random variables. Thus $E[J_i] = 1/p$ and $\text{Var}[J_i] = (1-p)/p^2$. Thus $\mathbf{J}$ has expected value

$$E[\mathbf{J}] = \boldsymbol{\mu}_J = \begin{bmatrix} E[J_1] & E[J_2] & E[J_3] \end{bmatrix}' = \begin{bmatrix} 1/p & 1/p & 1/p \end{bmatrix}'. \tag{2}$$

Since the components of $\mathbf{J}$ are independent, it has the diagonal covariance matrix

$$\mathbf{C}_J = \begin{bmatrix} \text{Var}[J_1] & 0 & 0 \\ 0 & \text{Var}[J_2] & 0 \\ 0 & 0 & \text{Var}[J_3] \end{bmatrix} = \frac{1-p}{p^2}\mathbf{I} \tag{3}$$

Given these properties of $\mathbf{J}$, finding the same properties of $\mathbf{K} = \mathbf{A}\mathbf{J}$ is simple.

(a) The expected value of $\mathbf{K}$ is

$$E[\mathbf{K}] = \mathbf{A}\boldsymbol{\mu}_J = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1/p\\ 1/p\\ 1/p\end{bmatrix} = \begin{bmatrix}1/p\\ 2/p\\ 3/p\end{bmatrix} \tag{4}$$

(b) From Theorem 5.13, the covariance matrix of $\mathbf{K}$ is

$$\mathbf{C}_K = \mathbf{A}\mathbf{C}_J\mathbf{A}' \tag{5}$$

$$= \frac{1-p}{p^2}\mathbf{A}\mathbf{I}\mathbf{A}' \tag{6}$$

$$= \frac{1-p}{p^2}\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} = \frac{1-p}{p^2}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix} \tag{7}$$

(c) Given the expected value vector $\boldsymbol{\mu}_K$ and the covariance matrix $\mathbf{C}_K$, we can use Theorem 5.12 to find the correlation matrix

$$\mathbf{R}_K = \mathbf{C}_K + \boldsymbol{\mu}_K\boldsymbol{\mu}_K' \tag{8}$$

$$= \frac{1-p}{p^2}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix} + \begin{bmatrix}1/p\\ 2/p\\ 3/p\end{bmatrix}\begin{bmatrix}1/p & 2/p & 3/p\end{bmatrix} \tag{9}$$

$$= \frac{1-p}{p^2}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix} + \frac{1}{p^2}\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix} \tag{10}$$

$$= \frac{1}{p^2}\begin{bmatrix}2-p & 3-p & 4-p\\ 3-p & 6-2p & 8-2p\\ 4-p & 8-2p & 12-3p\end{bmatrix} \tag{11}$$

Problem 5.6.9 Solution

Given an arbitrary random vector $\mathbf{X}$, we can define $\mathbf{Y} = \mathbf{X} - \boldsymbol{\mu}_\mathbf{X}$ so that

$$\mathbf{C}_\mathbf{X} = E\left[(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})'\right] = E\left[\mathbf{Y}\mathbf{Y}'\right] = \mathbf{R}_\mathbf{Y}. \tag{1}$$

It follows that the covariance matrix $\mathbf{C}_\mathbf{X}$ is positive semi-definite if and only if the correlation matrix $\mathbf{R}_\mathbf{Y}$ is positive semi-definite. Thus, it is sufficient to show that every correlation matrix, whether it is denoted $\mathbf{R}_\mathbf{Y}$ or $\mathbf{R}_\mathbf{X}$, is positive semi-definite.

To show a correlation matrix $\mathbf{R}_\mathbf{X}$ is positive semi-definite, we write

$$\mathbf{a}'\mathbf{R}_\mathbf{X}\mathbf{a} = \mathbf{a}'E\left[\mathbf{X}\mathbf{X}'\right]\mathbf{a} = E\left[\mathbf{a}'\mathbf{X}\mathbf{X}'\mathbf{a}\right] = E\left[(\mathbf{a}'\mathbf{X})(\mathbf{X}'\mathbf{a})\right] = E\left[(\mathbf{a}'\mathbf{X})^2\right]. \tag{2}$$

We note that $W = \mathbf{a}'\mathbf{X}$ is a random variable. Since $E[W^2] \geq 0$ for any random variable W,

$$\mathbf{a}'\mathbf{R}_\mathbf{X}\mathbf{a} = E\left[W^2\right] \geq 0. \tag{3}$$