Rutgers, The State University Of New Jersey

Department of Electrical and Computer Engineering

332:541 **Stochastic Signals and Systems** Fall 2006

Assigned Problem 1

Due Wednesday, October 11, 2006

Problem 1 Let $X_1, X_2, \ldots, X_n$ be a collection of iid random variables each with CDF $F_{X_i}(x) = F_X(x)$ and PDF $f_{X_i}(x) = f_X(x)$. Let $Y_1, Y_2, \ldots, Y_n$ be defined by

$$Y_1 = X_1, \qquad Y_2 = \max(X_1, X_2), \qquad \ldots \qquad Y_n = \max(X_1, X_2, \ldots, X_n).$$

(a) Find the joint CDF $F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$.

(b) Find the joint PDF $f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$.

Comments: The random variables $Y_1, \ldots, Y_n$ are dependent since $Y_i = \max(Y_{i-1}, X_i)$. Also, if you use the solution to part (a) in a careless way, you are likely to get the wrong answer for part (b).

Solution

(a) We start by finding the joint CDF

$$\begin{aligned} & F_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) \\ & \quad = P\{Y_1 \le y_1, Y_2 \le y_2, \ldots, Y_n \le y_n\} && (1) \\ & \quad = P\{X_1 \le y_1, \max(X_1, X_2) \le y_2, \ldots, \max(X_1, \ldots, X_n) \le y_n\}. && (2) \end{aligned}$$

Since

$$\{\max(X_1, X_2, \ldots, X_i) \le y_i\} = \{X_1 \le y_i, X_2 \le y_i, \ldots, X_i \le y_i\}, \tag{3}$$

we have that

$$\begin{aligned} & F_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) \\ & \quad = P\{X_1 \le y_1, X_1 \le y_2, X_2 \le y_2, \ldots, X_1 \le y_n, X_2 \le y_n, \ldots, X_n \le y_n\} && (4) \\ & \quad = P\{X_1 \le \min(y_1, \ldots, y_n), X_2 \le \min(y_2, \ldots, y_n), \ldots, X_n \le y_n\}. && (5) \end{aligned}$$

Since the X_i are iid, we can conclude that

$$\begin{aligned} & F_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) \\ & \quad = P\{X_1 \le \min(y_1, \ldots, y_n)\} P\{X_2 \le \min(y_2, \ldots, y_n)\} \cdots P\{X_n \le y_n\} && (6) \\ & \quad = F_X(\min(y_1, \ldots, y_n)) F_X(\min(y_2, \ldots, y_n)) \cdots F_X(y_n) && (7) \\ & \quad = \prod_{i=1}^{n} F_X(\min(y_i, \ldots, y_n)). && (8) \end{aligned}$$

This is the correct answer!

Since $Y_i = \max(Y_{i-1}, X_i)$, we observe that $Y_i \ge Y_{i-1}$ and thus $Y_1 \le Y_2 \le \cdots \le Y_n$. As a result, it is tempting (but wrong) to conclude that that we only need to define the CDF $F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$ for $y_1 \le y_2 \le \cdots \le y_n$. In fact, it is important to keep in mind that the CDF must be correctly formulated for all values of $y_1, \ldots, y_n$.

(b) Usually it is simple to find the PDF from the CDF. This problem is different in that it is simple for certain values of $y_1, \ldots, y_n$ but not others. The simplest case to consider is

- $y_1 < y_2 < \cdots < y_n$:
  In this case, $\min(y_i, \ldots, y_n) = y_i$ and the joint CDF simplifies to

$$F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = F_X(y_1) F_X(y_2) \cdots F_X(y_n). \tag{9}$$

  In this case, the joint PDF also becomes simple:

$$\begin{aligned} f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) &= \frac{\partial^n F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)}{\partial y_1 \cdots \partial y_n} && (10)\\ &= \frac{\partial F_X(y_1)}{\partial y_1} \frac{\partial F_X(y_2)}{\partial y_2} \cdots \frac{\partial F_X(y_n)}{\partial y_n} && (11)\\ &= f_X(y_1) f_X(y_2) \cdots f_X(y_n). && (12) \end{aligned}$$

  This result is correct for $y_1 < y_2 < \cdots < y_n$.
  Since we know that $Y_1 \le Y_2 \le \cdots \le Y_n$, it it is tempting to but wrong to conclude the joint PDF is zero for values of $y_1, \ldots, y_n$ not satisfying $y_1 < y_2, \cdots < y_n$. As we see in the next case, this is correct in some cases.

- $y_{j+1} < y_j$ for some j:
  In this case, for any $i \le j$,

$$\min(y_i, \ldots, y_{j-1}, y_j, y_{j+1}, y_{j+2}, \ldots, y_n) = \min(y_i, \ldots, y_{j-1}, y_{j+1}, y_{j+2}, \ldots, y_n), \tag{13}$$

  which is not a function of y_j. Similarly for $i > j$, $\min(y_i, \ldots, y_n)$ is not a function of y_j. Thus,

$$\begin{aligned} &F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)\\ &\quad= \prod_{i=1}^{n} F_X(\min(y_i, \ldots, y_n)) && (14)\\ &\quad= \left(\prod_{i=1}^{j} F_X(\min(y_i, \ldots, y_n))\right) \left(\prod_{i=j+1}^{n} F_X(\min(y_i, \ldots, y_n))\right) && (15)\\ &\quad= \underbrace{\left(\prod_{i=1}^{j} F_X(\min(y_i, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n))\right)}_{\text{not a function of } y_j} \underbrace{\left(\prod_{i=j+1}^{n} F_X(\min(y_i, \ldots, y_n))\right)}_{\text{not a function of } y_j} && (16) \end{aligned}$$

  Hence $\partial F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) / \partial y_j = 0$, and we conclude that if $y_{j+1} < y_j$ for some j, then the joint PDF is zero.

This led a lot of people to conclude (alas by wrongly ignoring the boundary conditions where $y_j = y_{j+1}$) that the correct answer is

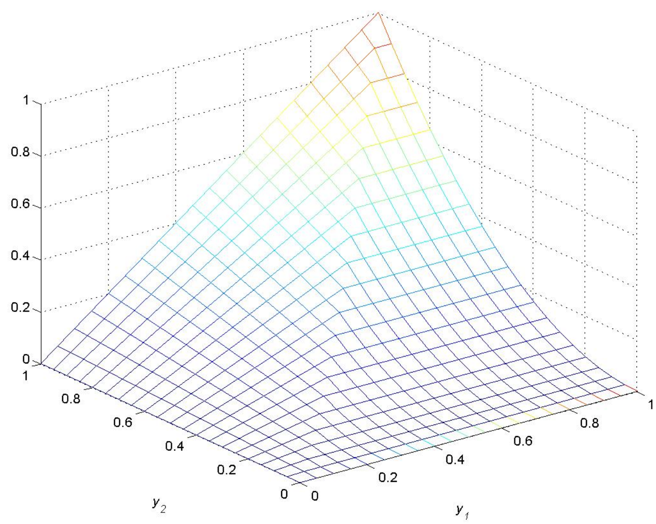
$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \begin{cases} f_X(y_1) \cdots f_X(y_n) & y_1 < y_2 < \cdots < y_n, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

However, it's easy to see that this result is wrong. Consider the case when the X_i are uniform $(0,1)$ random variables. In this case, $f_X(x) = 1$ for $0 \le x \le 1$ and we would obtain

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 < y_2 \le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

which fails to integrate to unity.

The problem is that the event $y_{j+1} = y_j$ occurs with nonzero probability. This can be seen by looking at a plot of the joint CDF where the derivative is not well defined along the line $y_1 = y_2$:



In particular, for $n = 2$, the PDF $f_{Y_1,Y_2}(y_1, y_2)$ isn't well defined along the ling $y_1 = y_2$. Given $Y_1 = y_1$, we can calculate the conditional PDF of Y_2 with the observation that

$$Y_2 = \begin{cases} y_1 & X_2 \le y_1 \\ X_2 & X_2 > y_1. \end{cases} \tag{19}$$

Thus, given $Y_1 = y_1$, $Y_2 = y_1$ with probability $P\{X_2 \le y_1\} = F_X(y_1)$. It's easy to show that given $Y_1 = y_1$, the conditional PDF of Y_2 is

$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} F_X(y_1)\,\delta(y_2 - y_1) + f_X(y_2) & y_2 \ge y_1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

$$= F_X(y_1)\,\delta(y_2 - y_1) + u(y_2 - y_1) f_X(y_2)\,. \tag{21}$$

where $u(x)$ denotes the unit step function. Since $Y_1 = X_1$, $f_{Y_1}(y_1) = f_X(y_1)$. It follows that the joint PDF is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)\, f_{Y_2|Y_1}(y_2|y_1) \tag{22}$$

$$= f_X(y_1)\left(F_X(y_1)\,\delta(y_2 - y_1) + u(y_2 - y_1) f_X(y_2)\right). \tag{23}$$

For a single random variable, the impulse in the PDF is well-understood as a placeholder for a probability mass. In thise case, one probably should be careful and simply avoid using the joint PDF.