

Degrees of Freedom of MIMO Cellular Networks: Two-Cell Three-User-Per-Cell Case

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Abstract—In this paper we investigate the spatially-normalized degrees of freedom (sDoF) of 2-cell, multiple-input multiple-output (MIMO) cellular networks with three users per cell having M antennas at each user and N antennas at each base-station. We characterize the optimal sDoF/user for all values of M and N and show that the optimal sDoF is a piecewise linear function, with either M or N being the bottleneck. We assume all channels to be generic, and establish achievability through linear transmit beamforming strategies. We introduce the notion of *packing ratio* that describes the interference footprint or shadow cast by a set of transmit beamformers. Through this notion, we reinterpret the alternating behavior of the optimal sDoF and attribute it to the availability of sets of transmit beamformers with certain packing ratios. We also derive a new DoF outer bound when $\frac{5}{9} \leq \frac{M}{N} \leq \frac{3}{4}$.

I. INTRODUCTION

Degrees of freedom (DoF) is a useful and tractable metric in understanding the role of interference in wireless networks. The degrees of freedom of wireless networks with same number of antennas at all nodes have been studied in [1]–[5]. In this case, since all nodes have the same number of antennas, schemes based on decomposition and asymptotic interference alignment can be used to achieve the optimal DoF. Establishing the optimal DoF for networks with different number of antennas at different nodes is more challenging. While the optimal DoF for the case of K -user MIMO interference channel with different number of antennas is studied in [6], [7], optimal DoF of MIMO cellular networks with different number of antennas have not been exactly characterized to the best of our knowledge. Some recent progress has been made in establishing outer bounds [8]–[10] and achievable schemes [10]–[14] that are optimal under specific conditions.

In this paper we study 2-cell MIMO cellular networks with three users per cell having M antennas per user and N antennas per base-station (BS). The 2-cell network with two users/cell has already been studied in [15]. However, adding even one additional user per cell can significantly alter the characteristics of the network as is clearly seen with the 3-user MIMO interference channel in [6]. Since we do not yet have a unifying theory that applies to the DoF of 2-cell networks with any number of users, studying networks with a small number of users can provide valuable insight for such a development.

In our work, we assume all channels to be generic and time varying. Further, similar in spirit to [6], we allow for spatial extensions of a given network and study the spatially-normalized DoF (sDoF) for all γ , where γ denotes the ratio M/N . Spatial extensions result in channels that are generic with

no additional structure — making them significantly easier to study as compared to time or frequency extensions.

There are three main approaches to establishing achievability of the optimal DoF of a network. These techniques include the asymptotic interference alignment (AIA) scheme [1], the rational dimensions framework [16] and transmit/receive beamformer design over finite time/frequency extensions [1], [6]. In this paper, the achievability of the optimal DoF is established through careful construction of linear transmit beamformers over finite space extensions.

In order to better understand the underlying structure of interference alignment, we formalize the concept of *packing ratio*. The packing ratio of a given set of beamformers is the ratio between the number of beamformers in the set and the number of dimensions these beamformers occupy at the interfering base-station (BS). Packing ratios are useful in determining the extent to which interference can be aligned at the interfering BS. For example, for the 2-cell, 3-user/cell MIMO cellular network, when $\gamma \leq 2/3$, the best possible packing ratio is 2 : 1, i.e., a set of two beamformers corresponding to two users aligns onto a single dimension at the interfering BS. This suggests that if we had sufficiently many such sets of beamformers, no more than $2N/3$ DoF/cell are possible. This in fact turns out to be a tight upper bound whenever $5/9 \leq \gamma \leq 2/3$. Further, it is easier to visualize the achievability of the optimal sDoF using linear beamforming through packing ratios. In addition, the exact cause for the alternating behavior of the optimal sDoF where either M or N is the bottleneck becomes apparent.

The DoF outer bounds are based on a set of bounds established in [17] for general cellular networks with G cells and K users per cell. For the network considered here, these bounds are shown to be tight for all γ except when $\gamma \in [\frac{5}{9}, \frac{3}{4}]$. For the case when $\gamma \in [\frac{5}{9}, \frac{3}{4}]$, we establish a new set of bounds that are derived based on a proof technique first developed in [6].

II. PRELIMINARIES

A. System Model

We consider two interfering cells with three users in each cell. Each user is assumed to have M antennas and each BS is assumed to have N antennas. We denote the uplink channel from the j^{th} user in the i^{th} cell to the k^{th} BS as the $N \times M$ matrix $\mathbf{H}_{(ij,k)}$ and assume all channels to be generic and time varying. In the uplink, the j^{th} user in the i^{th} cell is assumed to transmit the $M \times 1$ signal vector $\mathbf{x}_{ij}(t)$, which satisfies an average power constraint $\frac{1}{T} \sum_{t=1}^T E(\|\mathbf{x}_{ij}(t)\|^2) \leq \rho$. Thus, the

received signal at the k^{th} BS is given by

$$\mathbf{y}_k = \sum_{i \in \{1,2\}} \sum_{j \in \{1,2,3\}} \mathbf{H}_{(ij,k)} \mathbf{x}_{ij} + \mathbf{n}_k \quad (1)$$

where \mathbf{n}_k is the $N \times 1$ vector representing circular symmetric additive white Gaussian noise $\sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$. The received signal is defined similarly for the downlink. Since we consider two statistically identical cells, we use the relative indices i and \bar{i} when referring to the two cells.

B. Existing Results for General MIMO Cellular Networks

First, we restate relevant results from [17] for establishing the DoF outer bounds. We define a (G, K, M, N) cellular network to be a MIMO cellular network with G cells, K users per cell, M antennas per users and N antennas per BS.

Define the set $\mathcal{Q}_1 = \{2, 3, \dots, (G-1)K\}$ and the set $\mathcal{Q}_2 = \{\frac{1}{G-1}, \frac{1}{G-2}, \dots, 1\}$. The following theorem establishes an outer bound on the DoF of MIMO cellular networks.

Theorem 2.1 ([17]) *If a (G, K, M, N) cellular network satisfies $M/N \leq 1/q$, where $q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$, then $N/(K+q)$ is an outer bound on the DoF/user of that network. If instead, $M/N \geq 1/q$ for some $q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$, then $Mq/(K+q)$ is an outer bound on the DoF/user for that network.*

The proofs of these outer bounds are based on a result in [2], where MIMO wireless X networks with A transmitters and B receivers are considered.

Next, we state existing achievability results on the DoF of MIMO cellular networks using one-sided decomposition followed by the application of the asymptotic interference alignment scheme in [1].

Theorem 2.2 *For the (G, K, M, N) cellular network, using one-sided decomposition, $\frac{MN}{KM+N}$ DoF/user are achievable when $(G-1)KM \geq N$.*

One-sided decomposition refers to splitting the multi-antenna users in multiple independent single-antenna users. Subsequent to such a decomposition, the asymptotic interference alignment scheme of [1] is used to prove achievability. Note that this is an inner bound on the achievable DoF of the original network.

C. Spatially-normalized DoF

We restate the definition of spatially-normalized DoF as given in [6].

Definition 2.1 *Denoting the DoF/user of a (G, K, M, N) cellular network as $\text{DoF}(M, N)$, the spatially-normalized DoF/user is defined as*

$$\text{sDoF}(M, N) = \max_{q \in \mathbb{Z}^+} \frac{\text{DoF}(qM, qN)}{q}. \quad (2)$$

Analogous to frequency and time domain symbol extensions, the definition above allows us to permit extensions in space, i.e., adding antennas at the transmitters and receivers while maintaining the ratio M/N to be a constant. Unlike time or frequency extensions where the resulting channels are block diagonal, spatial extensions assume generic channels with no additional structure. The lack of any structure in the channel obtained through space extensions makes it significantly easier to analyze the network.

III. MAIN RESULTS

We now present the main results in this paper. Define the function $f_\omega(\cdot)$ as

$$f_\omega(M, N) = \max\left(\frac{N\omega}{3\omega+1}, \frac{M}{3\omega+1}\right), \quad (3)$$

where $\omega \geq 0$. Further, define the function $D(\cdot)$ to be

$$D(M, N) = \min\left(N, KM, f_{\frac{1}{3}}(M, N), f_{\frac{1}{2}}(M, N), f_{\frac{2}{3}}(M, N), f_1(M, N)\right). \quad (4)$$

The following theorem characterizes an outer bound on the DoF/user of the 2-cell 3-user/cell MIMO cellular network.

Theorem 3.1 *The DoF/user of a 2-cell, 3-user/cell MIMO cellular network with M antennas per user and N antennas per BS is bounded above by*

$$\text{DoF/user} \leq D(M, N) \quad (5)$$

Note that since this outer bound is linear in either M or N , this bound is invariant to spatial normalization and hence is also a bound on sDoF and not just DoF. The proof of this theorem is presented in Section V.

The next theorem characterizes the sDoF/user of a 2-cell, 3-user/cell MIMO cellular network.

Theorem 3.2 *The spatially-normalized DoF of a 2-cell, 3-user/cell cellular network with M antennas per user and N antennas per BS is given by*

$$\text{sDoF/user} = D(M, N) \quad (6)$$

This result states that when spatial-extensions are allowed, the outer bound presented in Theorem 3.1 is tight. The achievability part of Theorem 3.2 is based on linear beamforming. The details of the achievable scheme are presented in Section IV.

Fig. 1 captures the main results presented in the above theorems and plots sDoF/user normalized by N as a function of γ . Just as in the 3-user interference channel [6], there is an alternating behavior in the sDoF with either M or N being the bottleneck for a given γ .

Fig. 1 also plots the boundary separating proper systems from improper systems. Since designing transmit and receive beamformers for linear interference alignment is equivalent to solving a system of bilinear equations, such systems are classified as being proper or improper based on whether the total number of variables exceeds the total number of constraints in the system of equations or not [18]. Suppose d DoF are desired per user in a (G, K, M, N) cellular network, the network is said to be proper if it satisfies $\frac{1+\gamma}{GK+1} \geq \frac{d}{N}$. Substituting $G=2$ and $K=2$, a 2-cell 3-users/cell network is proper iff $\frac{1+\gamma}{7} \geq \frac{d}{N}$. It is seen from Fig. 1 that not all proper systems are feasible. In fact, systems with $\gamma \in \{1/6, 2/5, 5/9, 3/4, 4/3\}$ are the only ones that lie on the boundary between proper and improper systems and are feasible.

From Fig. 1, we can see that when $M/N \in \{1/6, 2/5, 5/9, 3/4, 4/3\}$, neither M nor N has any redundant dimensions, and decreasing either of them affects the sDoF.

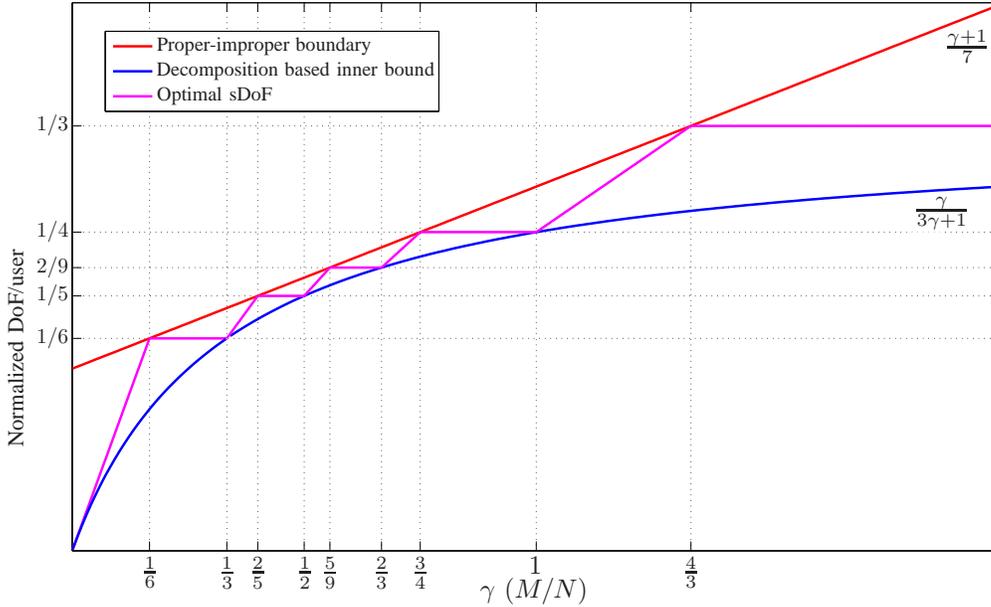


Fig. 1: The sDoF/user (normalized by N) of a 2-cell, 3-user/cell MIMO cellular network as a function of γ .

On the other hand, when $M/N \in \{1/3, 1/2, 2/3, 1\}$, both M and N have redundant dimensions, and some dimensions from either M or N can be sacrificed without losing any sDoF. For all other cases, only one of M or N is a bottleneck.

Fig. 1 also plots the achievable DoF by first decomposing the multi-antenna users into single-antenna users followed by using the asymptotic alignment scheme of [1]. Interestingly, the only cases where the decomposition based inner bound achieves the optimal sDoF is when the both M and N have redundant dimensions i.e., $M/N \in \{1/3, 1/2, 2/3, 1\}$.

IV. ACHIEVABILITY OF THE OPTIMAL SDOF

We now present the linear transmit beamforming strategy that we use to achieve the optimal sDoF of the 2-cell 3-user/cell MIMO cellular network. We consider achievability only in the uplink as duality of interference alignment through linear beamforming ensures achievability in the downlink as well. We start by introducing a new notion called the *packing ratio* to describe a collection of transmit beamforming vectors.

Definition 4.1 Consider the uplink of a 2-cell network and let \mathcal{S} be a collection of transmit beamformers used by users belonging to the same cell. If the number of dimensions occupied by the signals transmitted using this set of beamformers at the interfering BS is denoted by d , then the packing ratio η of this set of beamformers is given by $|\mathcal{S}|:d$.

As an example, consider a 2-cell, 3-user/cell cellular network with 2 antennas at each user and 3 antennas at each BS. Suppose we design two beamformers \mathbf{v} and \mathbf{w} for two different users in the same cell so that $\mathbf{H}_{11,2}\mathbf{v} = \mathbf{H}_{12,2}\mathbf{w}$, then the set of vectors $\mathcal{S} = \{\mathbf{v}, \mathbf{w}\}$ is said to have a packing ratio of 2:1. As another example, for the same network, consider the case when $M > N$. Since users can now zero-force all antennas at the interfering

BS, we can have a set \mathcal{S} of beamformers with packing ratio $|\mathcal{S}|:0$.

When designing beamformers for the 2-cell network, it is clear that choosing sets of beamformers having a high packing ratio is desirable as this reduces the number of dimensions occupied by interference at the interfering BS. The existence of beamformers satisfying a certain packing ratio is closely related to the ratio $\gamma (M/N)$. For example, it is easily seen that when $\gamma < \frac{2}{3}$, it is not possible to construct beamformers having a packing ratio of 3:1. Further even when beamformers satisfying a certain packing ratio exist, there may not be sufficient sets of them to completely use all the available dimensions at a BS. In such a scenario, we need to consider designing beamformers with the next best packing ratio.

Using the notion of packing ratios, we now describe the achievability of the optimal DoF of the 2-cell 3-user/cell cellular network. We first define the set $\mathcal{P} = \{1:0, 3:1, 2:1, 3:2, 1:1\}$ to be the set of fundamental packing ratios for the 2-cell, 3-user/cell cellular network. For any given γ , our strategy is to first construct the sets of beamformers that have the highest possible packing ratio from the set \mathcal{P} . If such beamformers do not completely utilize all the available dimensions at the two BSs, we further construct beamformers having the next best packing ratio in \mathcal{P} until all the dimensions at the two BSs are either occupied by signal or interference. This is illustrated in the following.

We first consider the case when $2/3 < \gamma < 3/4$. Note that since $M < N$, no transmit zero-forcing is possible. Further, note that each user can access only M of the N dimensions at the interfering BS. Since we assumed all channels to be generic, and $2M > N$, the subspaces accessible to any two users overlap in $2M - N$ dimensions. This $2M - N$ dimensional space overlaps with the M dimensions accessible to the third user in $3M - 2N$ dimensions. Note that such a space exists as we

TABLE I: The sets of beamformers and their corresponding packing ratios used to prove achievability of the sDoF for different values of γ .

γ	Set of beamformers				# signal-vectors/cell (DoF/cell)
	Packing ratio	# sets	Packing ratio	# sets	
$0 < \gamma < \frac{1}{6}$	1:1	$3M$	–	–	$3M$
$\frac{1}{6} \leq \gamma < \frac{1}{3}$	1:1	$\frac{N}{2}$	–	–	$\frac{N}{2}$
$\frac{1}{3} \leq \gamma < \frac{2}{5}$	3:2	$3M - N$	1:1	$\frac{6N-15M}{2}$	$\frac{3M}{2}$
$\frac{2}{5} \leq \gamma < \frac{1}{2}$	3:2	$\frac{N}{5}$	–	–	$\frac{3N}{5}$
$\frac{1}{2} \leq \gamma < \frac{5}{9}$	2:1	$3(2M - N)$	3:2	$\frac{10N-18M}{5}$	$\frac{6M}{5}$
$\frac{5}{9} \leq \gamma < \frac{2}{3}$	2:1	$\frac{N}{3}$	–	–	$\frac{2N}{3}$
$\frac{2}{3} \leq \gamma < \frac{3}{4}$	3:1	$3M - 2N$	2:1	$3N - 4M$	M
$\frac{3}{4} \leq \gamma < 1$	3:1	$\frac{N}{4}$	–	–	$\frac{3N}{4}$
$1 \leq \gamma < \frac{4}{3}$	1:0	$3(M - N)$	3:1	$N - \frac{3M}{4}$	$\frac{3M}{4}$
$\frac{4}{3} \leq \gamma$	1:0	N	–	–	N

have assumed $2/3 < \gamma$. Thus, we can construct $3M - 2N$ sets of three beamformers (one for each user) that occupy just one dimension at the interfering BS and thus have a packing ratio of 3:1. Assuming the same strategy is adopted for users in both cells, at any BS, signal vectors occupy a total of $3(3M - 2N)$ dimensions while interference occupies $3M - 2N$ dimensions. Thus a total of $4(3M - 2N)$ dimensions are occupied by signal and interference. Since $4(3M - 2N) < N$ whenever $4M < 3N$, we see that such vectors do not completely utilize all the N dimensions at a BS.

In order to utilize the remaining $9N - 12M$ dimensions, we additionally construct beamformers with the next highest packing ratio (2:1). Let $M' = M - (3M - 2N) = 2N - 2M$ denote the unused dimensions at each user. At the interfering BS, each pair of users has $2M' - N$ dimensions that can be accessed by both users. Note that since $2M' - N = 2(2N - 2M) - N = 3N - 2M > 0$, such an overlap exists almost surely. For a fixed pair of users in each cell, we choose $(3N - 4M)$ sets of two beamformers (one for each user in the pair) whose interference aligns onto a single dimension, so that each set has a packing ratio of 2:1. After choosing beamformers in this manner, we see that signal and interference span all N dimensions at each of the two BSs. Through this process, each BS receives $3(3M - 2N) + 2(3N - 4M)$ signaling vectors while interfering signals occupy $(3M - 2N) + (3N - 4M)$ dimensions. We have thus shown that $3(3M - 2N) + 2(3N - 4M) = M$ DoF/cell are achievable. An averaging argument ensures that $M/3$ DoF/user are achieved.

When $3/4 \leq \gamma < 1$, all three users of a cell can access a $3M - 2N$ dimensional space at the interfering BS, thus $3M - 2N$ sets of three beamformers having a packing ratio of 3:1 are possible. Note that 3:1 is still the highest possible packing ratio. If users in both cells were to use such beamformers, signal and interference from such beamformers can occupy at most $4(3M - 2N) > N$ dimensions at any BS. Thus, when $3/4 \leq \gamma < 1$, we have sufficient sets of beamformers with packing ratio 3:1 to use all available dimensions at the BSs. Choosing $N/4$ such sets provides us with $3N/4$ DoF/cell.

Such an approach to designing the linear beamformers provides insight on why the optimal sDoF alternates between M and N . When γ is such that there are sufficient sets of beamformers having the highest possible packing ratio in \mathcal{P} ,

it is the number of dimensions at the BSs that proves to be a bottleneck and the DoF bound becomes dependent on N . On the other hand, when there are not enough sets of beamformers having the highest possible packing ratio in \mathcal{P} , we are forced to design beamformers with a lower packing ratio so as to use all available dimensions at the two BSs. Since, for a fixed N , the number of sets of beamformers having the highest packing ratio is a function of M , the bottleneck now shifts to M . We thus see that for a large but fixed N , as we gradually increase M , we cycle through two stages—the first stage where beamformers with a higher packing ratio become feasible but are limited to a small number, and then gradually, the second stage where there are sufficiently many such beamformers. As M is increased even further, we go back to the scenario where the next higher packing ratio becomes feasible however with only limited set of beamformers, and so on.

The design strategy described for $2/3 < \gamma \leq 1$ is applicable to other intervals of γ as well. When $1/3 < \gamma \leq 1/2$, we design as many sets of beamformers having packing ratio 3:2 as possible and then use beamformers having a packing ratio of 1:1 (random beamforming) to fill any unused dimensions at the two BSs. When $1/2 < \gamma \leq 2/3$ we first design as many sets of beamformers having packing ratio 2:1 as possible and then use beamformers having a packing ratio of 3:2. When $\gamma \leq 1/3$, it is easy to see that simple zero-forcing strategy suffices. Finally, when $\gamma \geq 1$, we first design beamformers that zero-force the interfering BS (packing ratio 1:0), and then use beamformers having a packing ratio of 3:1 to fill any remaining dimensions at each BS. In order to keep the presentation short, we do not go into the exact details for these cases. In Table I, we summarize the strategies used for different intervals of γ , and list the number of sets of beamformers of a certain packing ratio required to achieve the optimal DoF along with the DoF achieved per cell. Note that fractional number of sets can always be made into integers as we allow for spatial extensions.

V. OUTER BOUND ON DOF

The outer bound on the DoF can be categorized into three cases.

Case i: ($0 \leq \gamma \leq 1/6$) & ($4/3 \leq \gamma$): The outer bounds in this case follow by letting the two BSs cooperate and then

considering the DoF bounds for the multiple-access channel and the broadcast channel.

Case ii: ($1/6 \leq \gamma \leq 5/9$) & ($3/4 \leq \gamma \leq 4/3$): The bounds in this case are derived from Theorem 2.1. Specifically, we consider $q = 3$ for $1/6 \leq \gamma \leq 2/5$, $q = 2$ for $2/5 \leq \gamma \leq 5/9$, and $q = 1$ for $3/4 \leq \gamma \leq 4/3$.

Case iii: ($5/9 \leq \gamma \leq 3/4$): We derive a new set of outer bounds on the DoF for this case. Our approach to deriving these new bounds is inspired by the approach taken in [6]. The exact details of this derivation are presented in the following section.

Note that since the outer bounds scale linearly in the transmit/receive antennas, these are also bounds on the sDoF/user of this network.

A. DoF Outer Bound when $\frac{5}{9} \leq \gamma < \frac{3}{4}$

In this section, we show that whenever $\frac{5}{9} \leq \gamma \leq \frac{3}{4}$, no more than $\max(\frac{2N}{9}, \frac{M}{3})$ DoF/user are possible. Since there is no duality associated with the information theoretic proof presented here, we need to establish this result separately for uplink and downlink. Due to space constraints, we only present the proof for the uplink. Similar to [6], we first perform an invertible linear transformation at the users and the base-stations. The linear transformation involves multiplication by a full rank matrix at each user and BS. Let the $M \times M$ transformation matrix at user ij be denoted as \mathbf{U}_{ij} and the $N \times N$ transformation matrix at BS \bar{i} be denoted as $\mathbf{B}_{\bar{i}}$. Using these transformations the effective channel between user ij and BS \bar{i} is given by $\mathbf{B}_{\bar{i}}\mathbf{H}_{(ij,\bar{i})}\mathbf{U}_{ij}$. Subsequent to this transformation, we identify genie signals that enable the BSs to decode all the messages in the network and set up a bound on the sum-rate of the network. We start by considering the case when $5/9 \leq \gamma \leq 2/3$.

1) *DoF outer bound when $5/9 \leq \gamma \leq 2/3$:* We divide the set of N antennas at BS \bar{i} into three groups and denote them as \bar{ia} , \bar{ib} and \bar{ic} . The sets \bar{ia} and \bar{ic} contain the first and last $N - M$ antennas each while set \bar{ib} has the remaining $2M - N$ antennas. Let the M antennas at user ij be denoted as ijk where $k \in \{1, 2, \dots, M\}$. Using a similar notation for BS antennas, let $\mathbf{H}_{(ij,\bar{ip}:\bar{iq})}$ represent the channel from user ij to the subset of BS antennas from the p^{th} antenna to the q^{th} antenna.

We first focus on the $N \times M$ channel from user $i1$ to BS \bar{i} . We set the first $N - M$ rows of $\mathbf{B}_{\bar{i}}$ to be orthogonal to the columns of $\mathbf{H}_{(i1,\bar{i})}$. Since $\mathbf{H}_{(i1,\bar{i})}$ spans only M of the N dimensions at BS \bar{i} , it is possible to choose such a set of vectors. Similarly, the next $2M - N$ and $N - M$ rows of $\mathbf{B}_{\bar{i}}$ are chosen to be orthogonal to user $i2$ and user $i3$ respectively. Since all channels are assumed to be generic, matrix $\mathbf{B}_{\bar{i}}$ is guaranteed to be full rank almost surely.

On the user side, user $i1$ inverts the channel to the last M antennas of BS \bar{i} , i.e., $\mathbf{U}_{i1} = (\mathbf{H}_{(i1,\bar{i}N-M+1:\bar{i}N)})^{-1}$, while user $i3$ inverts the channel to the first M antennas of BS \bar{i} , i.e., $\mathbf{U}_{i3} = (\mathbf{H}_{(i1,\bar{i}1:\bar{i}M)})^{-1}$. We let $\mathbf{U}_{i2} = \mathbf{I}$. The signal structure resulting from such a transformation is shown in Fig. 2.

Let m_{ij} be the message from user ij to BS \bar{i} . This message is mapped to a $M \times n$ codeword $\mathbf{x}_{ij}^n = \{\mathbf{x}_{ijk}^n : k \in (1, 2, \dots, M)\}$, where n is the length of the code. We denote the rate to user ij as R_{ij} , the total sum rate of the network as R_{sum} and the collection of all messages in the network as $\{m_{ij}\}$.

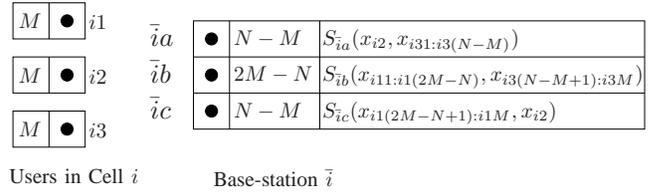


Fig. 2: The signal structure obtained after linear transformation for the case when $\gamma \leq 2/3$. Note that the figure does not include signals from the same cell.

Now, consider providing the set of signals $\mathcal{G}_1 = \{\tilde{\mathbf{x}}_{i2}^n, \tilde{\mathbf{x}}_{i11:i1(2M-N)}^n\}$ to BS \bar{i} . We use $\tilde{\mathbf{x}}^n$ to denote $\mathbf{x}^n + \mathbf{z}^n$ where \mathbf{z}^n is circular symmetric Gaussian noise that is artificially added to the transmitted signal \mathbf{x}^n . Since we seek to establish a converse, we assume that BS \bar{i} can decode all the messages from its users. After decoding and subtracting these signals from the received signal, the resulting signals at the three antenna sets are given in Fig. 2 where $S_{\bar{i}^*}(\cdot)$ represents a noisy linear combination of its arguments. Given \mathcal{G}_1 , we can subtract $\tilde{\mathbf{x}}_{i2}^n$ from $S_{\bar{ic}}(x_{i1(2M-N+1):i1M}, x_{i2})$ and along with $\tilde{\mathbf{x}}_{i11:i1(2M-N)}^n$ from \mathcal{G}_1 , we can decode m_{i1} subject to noise distortion. After decoding m_{i1} , and subtracting $\tilde{\mathbf{x}}_{i1}^n$ and $\tilde{\mathbf{x}}_{i2}^n$ from the received signal, m_{i3} can also be decoded subject to noise distortion. Since BS \bar{i} can recover all the messages in the network given $\mathbf{y}_{\bar{i}}^n$ and \mathcal{G}_1 subject to noise distortion, we have

$$\begin{aligned}
nR_{sum} &\leq I(\{m_{ij}\}; \mathbf{y}_{\bar{i}}^n, \mathcal{G}_1) + n o(\log \rho) + o(n) \\
&\stackrel{a}{\leq} Nn \log \rho + h(\tilde{\mathbf{x}}_{i2}^n, \tilde{\mathbf{x}}_{i11:i1(2M-N)}^n | \mathbf{y}_{\bar{i}}^n) + n o(\log \rho) + o(n) \\
&\stackrel{b}{\leq} Nn \log \rho + nR_{i2} + h(\tilde{\mathbf{x}}_{i11:i1(2M-N)}^n) + n o(\log \rho) + o(n) \\
&\stackrel{c}{\leq} Nn \log \rho + nR_{i2} + h(\tilde{\mathbf{x}}_{i11:i1(2M-N)}^n) + n o(\log \rho) + o(n)
\end{aligned} \tag{7}$$

where (a) follows from Fano's inequality, (b) follows from Lemma 3 in [6] and (c) follows from the fact that conditioning reduces entropy.

Next, consider providing the set of signals $\mathcal{G}_2 = \{\tilde{\mathbf{x}}_{i3}^n, \tilde{\mathbf{x}}_{i1(2M-N+1):i1M}^n\}$ to BS \bar{i} . After subtracting $\tilde{\mathbf{x}}_{i3}^n$ from the received signal, the BS can recover m_{i2} from observations at antenna sets \bar{ia} and \bar{ic} subject to noise distortion. Subsequently, BS \bar{i} can also recover m_{i1} subject to noise distortion. Since BS \bar{i} can recover all messages when provided with the genie signal \mathcal{G}_2 , using similar steps as before, we obtain

$$\begin{aligned}
nR_{sum} &\leq Nn \log \rho + nR_{i3} + nR_{i1} - h(\tilde{\mathbf{x}}_{i11:i1(2M-N)}^n) \\
&\quad + n o(\log \rho) + o(n)
\end{aligned} \tag{8}$$

Adding (7) and (8) we get,

$$2nR_{sum} \leq 2nN \log \rho + \sum_{j=1,2,3} nR_{ij} + n o(\log \rho) + o(n) \tag{9}$$

Using a similar inequality for BS i , we can write

$$3nR_{sum} \leq 4nN \log \rho + n o(\log \rho) + o(n) \tag{10}$$

Letting $n \rightarrow \infty$ and $\rho \rightarrow \infty$, we see that DoF/user $\leq \frac{2N}{9}$.

$N - M$	●	$i1a$		
$3M - 2N$	●	$i1b$		
$N - M$	●	$i1c$		
$N - M$	●	$i2a$	$\bar{i}a$	● $N - M$ $S_{\bar{i}a}(x_{i2b}, x_{i2c}, x_{i3c})$
$3M - 2N$	●	$i2b$	$\bar{i}b$	● $2M - N$ $S_{\bar{i}b}(x_{i1b}, x_{i1c}, x_{i3a}, x_{i3b})$
$N - M$	●	$i2c$	$\bar{i}c$	● $N - M$ $S_{\bar{i}c}(x_{i1a}, x_{i2a})$
$N - M$	●	$i3a$		
$3M - 2N$	●	$i3b$		
$N - M$	●	$i3c$		

Users in Cell i Base-station \bar{i}

Fig. 3: The signal structure obtained after linear transformation when $\gamma \geq 2/3$. The figure does not include signals from the same cell.

2) *DoF outer bound when $2/3 \leq \gamma \leq 3/4$* : In this case, we again group the antennas at BS \bar{i} into three groups exactly as before. The M antennas at each user are also grouped into three sets as shown in Fig. 3. The linear transformation at BS \bar{i} is also same as before, i.e., each group of antennas zero-forces one of three users.

On the user side, \mathbf{U}_{i1} for user $i1$ is chosen such that $i1a$ zero-forces $\bar{i}b$ while $i1b$ and $i1c$ both zero-force $\bar{i}c$. Similarly, \mathbf{U}_{i3} is chosen so that $i3c$ zero-forces $\bar{i}b$, while $i3b$ and $i3c$ both zero-force $\bar{i}a$ and finally \mathbf{U}_{i2} is chosen such that $i2a$ zero-forces $\bar{i}a$, while $i2b$ and $i2c$ both zero-force $\bar{i}c$. The resulting signal structure at BS \bar{i} after removing signals from Cell \bar{i} is given in Fig. 3.

Now, consider providing the set of signals $\mathcal{G}_3 = \{\tilde{\mathbf{x}}_{i1}^n, \tilde{\mathbf{x}}_{i2b}^n, \tilde{\mathbf{x}}_{i2c}^n\}$ to BS \bar{i} . After decoding the messages from users in Cell \bar{i} , we see that using \mathcal{G}_3 , we can first decode m_{i2} followed by m_{i3} , subject to noise distortion. Since BS \bar{i} can recover all the messages in the network given $\mathbf{y}_{\bar{i}}^n$ and \mathcal{G}_3 , subject to noise distortion, we have

$$\begin{aligned}
nR_{sum} &\leq I(\{m_{ij}\}; \mathbf{y}_{\bar{i}}^n, \mathcal{G}_1) + n o(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{i1}^n, \tilde{\mathbf{x}}_{i2b}^n, \tilde{\mathbf{x}}_{i2c}^n | \mathbf{y}_{\bar{i}}^n) + n o(\log \rho) + o(n) \\
&\leq Nn \log \rho + nR_{i1} + h(\tilde{\mathbf{x}}_{i2b}^n, \tilde{\mathbf{x}}_{i2c}^n | \tilde{\mathbf{x}}_{i2a}^n) + n o(\log \rho) + o(n) \\
&\leq Nn \log \rho + nR_{i1} + nR_{i2} - h(\tilde{\mathbf{x}}_{i2a}^n) + n o(\log \rho) + o(n), \tag{11}
\end{aligned}$$

where $\tilde{\mathbf{x}}_{i2a}^n$ denotes \mathbf{x}_{i2a}^n corrupted by channel noise.

Next, we consider the genie signal $\mathcal{G}_4 = \{\tilde{\mathbf{x}}_{i3}^n, \tilde{\mathbf{x}}_{i2a}^n, \tilde{\mathbf{x}}_{i2b}^n\}$. It can once again be shown that BS \bar{i} can recover all the messages in the network given $\mathbf{y}_{\bar{i}}^n$ and \mathcal{G}_4 . Going through similar steps as before, it can be shown that

$$\begin{aligned}
nR_{sum} &\leq (3M - N)n \log \rho + nR_{i3} + h(\tilde{\mathbf{x}}_{i2a}^n) \\
&\quad + n o(\log \rho) + o(n) \tag{12}
\end{aligned}$$

Adding (11) and (12), we get

$$2nR_{sum} \leq 3Mn \log \rho + \sum_{j=1,2,3} nR_{ij} + n o(\log \rho) + o(n) \tag{13}$$

By symmetry we must also have an analogous inequality involving the rates $R_{\bar{i}j}$, and adding these two inequalities, we get

$$3nR_{sum} \leq 6Mn \log \rho + n o(\log \rho) + o(n) \tag{14}$$

Letting $n \rightarrow \infty$ and $\rho \rightarrow \infty$, we see that DoF/user $\leq \frac{M}{3}$.

VI. CONCLUSION

This paper studies the DoF of a 2-cell, 3-users/cell network with M antennas at each user and N antennas at each BS. The achievability is established through linear transmit beamforming and finite spatial extensions. We formalize a concept called packing ratio that provides insight on achievability through linear beamforming and on the alternating behavior of the DoF outer bound. We also derive a new DoF outer bound when $5/9 \leq \gamma \leq 3/4$.

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