Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers

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Problem Solution : Yates and Goodman,3.5.3

Problem 3.5.3

First we observe that for $n = 1, 2, \ldots$, the marginal PMF of N satisfies

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = (1-p)^{n-1} p \sum_{k=1}^{n} \frac{1}{n} = (1-p)^{n-1} p$$

Thus, the event B has probability

$$P[B] = \sum_{n=10}^{\infty} P_N(n) = (1-p)^9 p[1 + (1-p) + (1-p)^2 + \cdots] = (1-p)^9$$

From Theorem 3.11,

$$\begin{aligned} P_{N,K|B}(n,k) &= \begin{cases} \frac{P_{N,K}(n,k)}{P[B]} & n,k \in B \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (1-p)^{n-10} p/n & n = 10, 11, \ldots; k = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The conditional PMF $P_{N|B}(n|b)$ could be found directly from $P_N(n)$ using Theorem 2.19. However, we can also find it just by summing the conditional joint PMF.

$$P_{N|B}(n) = \sum_{k=1}^{n} P_{N,K|B}(n,k) = \begin{cases} (1-p)^{n-10} p & n = 10, 11, \ldots \\ 0 & \text{otherwise} \end{cases}$$

From the conditional PMF $P_{N|B}(n)$, we can calculate directly the conditional moments of N given B. Instead, however, we observe that given B, $N' = N - 9$ has a geometric PMF with mean $1/p$. That is, for $n = 1, 2, \ldots$,

$$P_{N'|B}(n) = P[N = n + 9|B] = P_{N|B}(n+9) = (1-p)^{n-1} p$$

Hence, given B, $N = N' + 9$ and we can calculate the conditional expectations

$$\begin{aligned} E[N|B] &= E\left[N' + 9|B\right] = E\left[N'|B\right] + 9 = 1/p + 9 \\ \mathrm{Var}[N|B] &= \mathrm{Var}\left[N' + 9|B\right] = \mathrm{Var}\left[N'|B\right] = (1-p)/p^2 \end{aligned}$$

Note that further along in the problem we will need $E\left[N^2|B\right]$ which we now calculate.

$$\begin{aligned} E\left[N^2|B\right] &= \mathrm{Var}[N|B] + (E[N|B])^2 \\ &= \frac{2}{p^2} + \frac{17}{p} + 81 \end{aligned}$$

For the conditional moments of K, we work directly with the conditional PMF $P_{N,K|B}(n,k)$.

$$E[K|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} k \frac{(1-p)^{n-10} p}{n} = \sum_{n=10}^{\infty} \frac{(1-p)^{n-10} p}{n} \sum_{k=1}^{n} k$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[K|B] = \sum_{n=1}^{\infty} \frac{n+1}{2} (1-p)^{n-1} p = \frac{1}{2} E[N+1|B] = \frac{1}{2p} + 5$$

We now can calculate the conditional expectation of the sum.

$$E[N+K|B] = E[N|B] + E[K|B] = 1/p + 9 + 1/(2p) + 5 = \frac{3}{2p} + 14$$

The conditional second moment of K is

$$E\left[K^2|B\right] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} k^2 \frac{(1-p)^{n-10} p}{n} = \sum_{n=10}^{\infty} \frac{(1-p)^{n-10} p}{n} \sum_{k=1}^{n} k^2$$

Using the identity $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, we obtain

$$E\left[K^2|B\right] = \sum_{n=10}^{\infty} \frac{(n+1)(2n+1)}{6} (1-p)^{n-10} p = \frac{1}{6} E[(N+1)(2N+1)|B]$$

Applying the values of $E[N|B]$ and $E\left[N^2|B\right]$ found above, we find that

$$E\left[K^2|B\right] = \frac{E\left[N^2|B\right]}{3} + \frac{E[N|B]}{2} + \frac{1}{6} = \frac{2}{3p^2} + \frac{37}{6p} + 31\frac{2}{3}$$

Thus, we can calculate the conditional variance of K.

$$\mathrm{Var}[K|B] = E\left[K^2|B\right] - (E[K|B])^2 = \frac{5}{12p^2} - \frac{7}{6p} + 6\frac{2}{3}$$

To find the conditional correlation of N and K,

$$E[NK|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} nk \frac{(1-p)^{n-10} p}{n} = \sum_{n=10}^{\infty} (1-p)^{n-1} p \sum_{k=1}^{n} k$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[NK|B] = \sum_{n=10}^{\infty} \frac{n(n+1)}{2} (1-p)^{n-10} p = \frac{1}{2} E[N(N+1)|B] = \frac{1}{p^2} + \frac{9}{p} + 45$$