RUTGERS

School of Engineering
Department of Electrical and Computer Engineering

332:421 **Wireless Communications Systems** **Fall 2010**

Quiz I

Write all answers on the printed sheet

1. *(60 points)* **Cora and the Alien Invasion:** Cora the Communications Engineer has been hired by NASA to investigate the possibility of an imminent alien invasion through measurements taken by the Voyager deep space probe as it pierces the heliosphere (extended solar neighborhood). Through top secret research, NASA has determined that the signal level S follows the following distributions:

$$f_{S|H_0}(s|H_0) = \lambda^2 s e^{-\lambda s}$$

and

$$f_{S|H_1}(s|H_1) = \lambda s e^{-\frac{1}{2}\lambda s^2}$$

where H_1 means the aliens are planning an invasion, and H_0 not. In both cases, $s \geq 0$. Your job is to help Cora design a decision box which takes the measurement S and produces a decision about whether the aliens are invading or not and does so with minimum probability of error.

(a) *(20 points)* Please carefully sketch the two conditional distributions for $\lambda = 1$.

SOLUTION: *See FIGURE 1.*

$$f_{S|H_1}(s) = s e^{-\frac{1}{2}s^2}$$
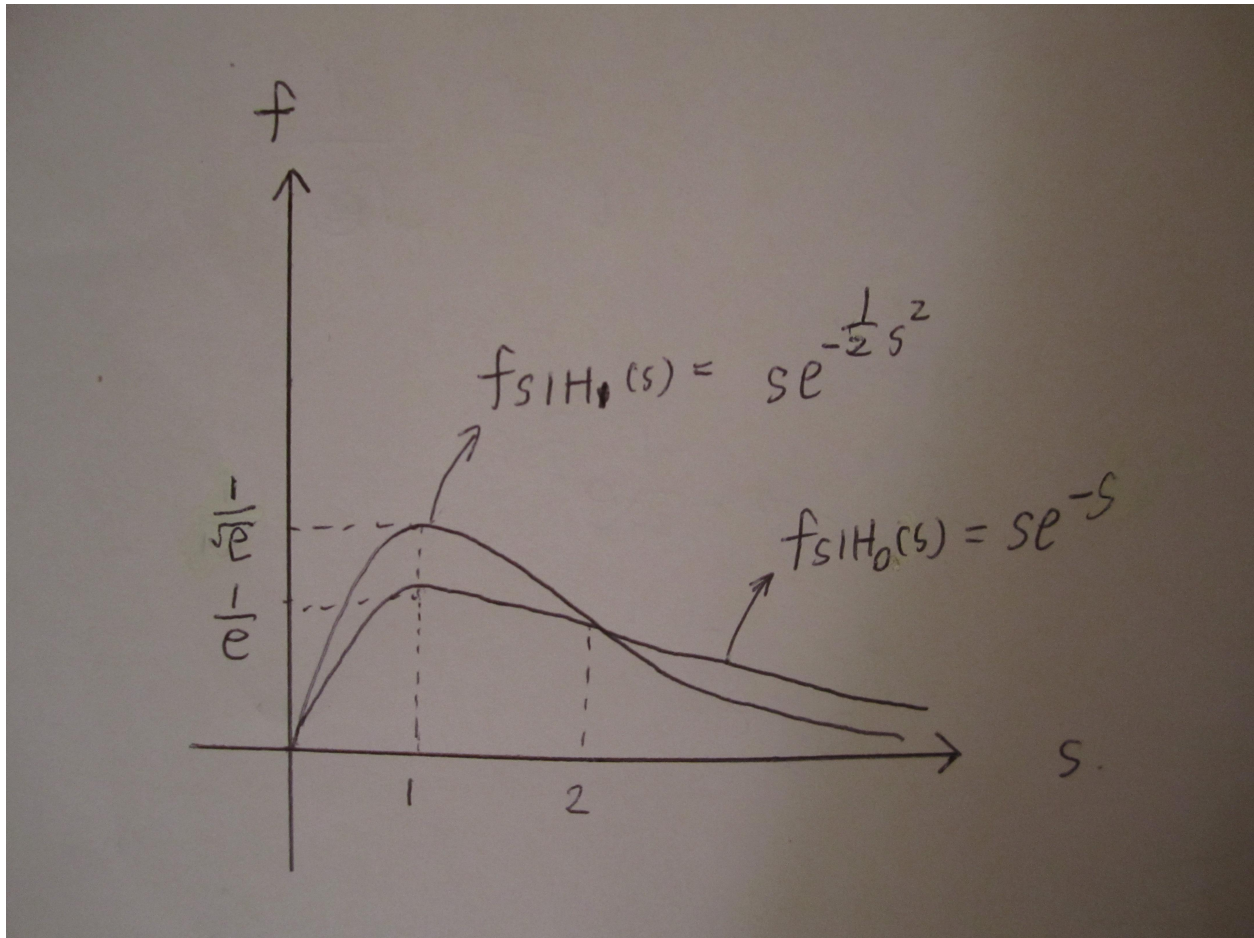
$$f_{S|H_0}(s) = s e^{-s}$$

Figure 1: Plot for Cora problem and $\lambda = 1$

(b) *(20 points)* If the aliens are planning an invasion with probability p, please provide an appropriate likelihood ratio for the decision regions associated with H_0 and H_1.

SOLUTION:

$$\frac{p f_{S|H_1}(s|H_1)}{(1-p) f_{S|H_0}(s|H_0)} = \frac{p \lambda s e^{-\frac{1}{2}\lambda s^2}}{(1-p)\lambda^2 s e^{-\lambda s}} = \frac{p e^{-\frac{1}{2}\lambda s^2}}{(1-p)\lambda e^{-\lambda s}} \mathop{\gtrless}_{No\ invasion}^{Invasion} 1$$

(c) *(20 points)* Please determine analytic expressions for decision regions for $p = 0.5$. What does your region reduce to if $\lambda = 1$?

SOLUTION: *For $p = 0.5$ we have*

$$\frac{e^{-\frac{1}{2}\lambda s^2}}{\lambda e^{-\lambda s}} = e^{-\frac{1}{2}\lambda s(s-2)} \quad \begin{array}{c} \textit{Invasion} \\ > \\ < \\ \textit{No invasion} \end{array} \quad \lambda$$

Since e^{-s^2} goes to zero much faster than e^{-s} and since the integral under each conditional distribution must be one (probability functions), we must have the H_1 conditional distribution larger over some region than the H_0 distribution.

So we seek to solve the equation for values of s where

$$e^{-\frac{1}{2}s(s-2)} = \lambda$$

Taking the log we have

$$-\frac{1}{2}\lambda s(s-2) = \log \lambda$$

so that

$$s^2 - 2s + a = 0$$

where $a = \frac{2}{\lambda} \log \lambda$. Therefore

$$s = 1 \pm \sqrt{1-a}$$

So on the interval $(1 - \sqrt{1-a}, 1 + \sqrt{1-a})$ we choose H_1.

For $\lambda = 1$ this reduces to $H_1 = \{s | s \in (0, 2)\}$

2. (*40 points*) **Signal Space:** A two-dimensional signal space uses basis functions $\phi_1(t) = 1$ and $\phi_2(t) = \sqrt{2} \cos 2\pi t$ on an interval $(0,1)$.

(a) (*10 points*) Please provide a signal space vector representation $\mathbf{s}_k$ for each of the functions $s_1(t) = \cos^2 \pi t$ and $s_2(t) = \sin^2 \pi t$ so that

$$s_k(t) = \mathbf{s}_k^\top \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = s_{k1}\phi_1(t) + s_{k2}\phi_2(t)$$

What is the energy in each signal? What is the distance between these two points in signal space?

SOLUTION: $\cos^2 \pi t = \frac{1}{2} + \frac{1}{2} \cos 2\pi t$ *and* $\sin^2 \pi t = \frac{1}{2} - \frac{1}{2} \cos 2\pi t$. *So* $\mathbf{s}_1 = [\frac{1}{2}, \frac{1}{2\sqrt{2}}]$ *and* $\mathbf{s}_2 = [\frac{1}{2}, -\frac{1}{2\sqrt{2}}]$.

The energy in each is the same and equal to $\frac{3}{8}$

The distance is $\sqrt{2}/2$.

(b) *(10 points)* On an interval $(0, 1)$ one of these signals is sent with equal probability. Zero mean white Gaussian noise of spectral height N_0, $w(t)$, is added to the transmission and a minimum probability of error receiver is used to decode whether signal 1 or signal 2 was sent. Under this scenario, the probability of error is some number P_e

Now, suppose we change the signal design and keep $s_1(t)$ as is but change $s_2(t) = -s_1(t)$. Does the total signal energy change? Does the probability of error go up or down? Why?

SOLUTION: *With white noise, the distance between the signal points matters. Under the replacement, the signal energies stay the same, but the distance increases to $\sqrt{3}$. Since the noise variances have not changed, this places the conditional Gaussian distributions which form the likelihood ratio farther apart in signal space and makes the integral under the tails (actually "skirts" since we're in 2-D) of the distributions smaller. So P_e goes down.*

(c) *(20 points)* Plot the two pairs of signal points in the same signal space coordinate frame and comment on binary signal design (where you have only two possible signals) under signal energy constraints in a multidimensional signal space. You may guess (with verbal justification) but to receive full credit you must PROVE your assertion.

SOLUTION: *When given an energy budget and only two signals, you should make them antipodal (opposites of each other) to maximize the distance and hence minimize the probability of error.*

Formally, we seek to maximize $d^2 = |\underline{s}_1 - \underline{s}_2|^2$ subject to a fixed energy constraint $E = |\underline{s}_1|^2 + |\underline{s}_2|$. First, we rewrite things as

$$d^2 = (\underline{s}_1 - \underline{s}_2)^\top (\underline{s}_1 - \underline{s}_2) = |\underline{s}_1|^2 + |\underline{s}_2|^2 - 2\underline{s}_1^\top \underline{s}_2 = E - 2\underline{s}_1^\top \underline{s}_2$$

Since we want to make d as large as possible, we need to make $\underline{s}_1^\top \underline{s}_2$ as negative as possible. Well, for fixed length vectors, we already know that the projection is maximized when both vectors are pointing in the same direction. So the dot product will be minized when they're pointing in opposite directions. The only thing left is to determine their amplitude. Let the amplitude of $\underline{s}_1$ be x and that of $\underline{s}_2$ be y. $E = x^2 + y^2$ so we can write $y = \sqrt{E - x^2}$. Then we have $\underline{s}_1^\top \underline{s}_2 = -xy = -xsqrtE - x^2$ which is maximized when $x = \sqrt{E/2}$.

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Don't believe me? We can just as easily maximize the square $x^2(E - x^2)$. Take the first derivative and set to zero: $2xE - 4x^3 = 0$ which is satisfied by $x = \frac{\sqrt{E}}{2}$ and $x = 0$.

3. *(50 points)* **Equalization** A communication channel has impulse response

$$h_k = (-a)^k$$

for $0 < a < 1$, $k = 0, 1, ...$ and is zero otherwise.

(a) *(25 points)* What two tap equalizer c_0, c_1 corrects the distortion introduced by h_k?
SOLUTION: *Take the z-transform of h_k, we get*

$$H(z) = 1/1 + az^{-1}$$

Because the equalizer is designed to equalize the effect of channel,

$$C(z) = 1/H(z) = 1 + az^{-1}$$

Take the inverse z-transform of C(z), we get

$$c_k = \delta(k) + a\delta(k-1)$$

Hence, $c_0 = 1$ and $c_1 = a$.

(b) *(10 points)* You are asked to build an adaptive equalizer for this system. Derive an expression for $\sum_{n=0}^{N} e_n^2$ where e_n is the difference between $\hat{x}_n$ the output of your two-tap equalizer filter and x_n.
SOLUTION:

$$\sum_{n=0}^{N} e_n^2 = \sum_{n=0}^{N} (\hat{x}_n - x_n)^2 = \sum_{n=0}^{N} (\sum_{n=0}^{1} c_k r_{n-k} - x_n)^2$$

$$= \sum_{n=0}^{N} (\sum_{n=0}^{1} c_k \sum_{l=0}^{n-k} x_l h_{n-k-l} - x_n)^2$$

(c) *(10 points)* Derive an explicit tap update equation for your adaptive equalizer using step size Δ.

SOLUTION:

$$\begin{bmatrix} c_0[m] \\ c_1[m] \end{bmatrix} - \Delta \, \nabla f \begin{bmatrix} c_0[m] \\ c_1[m] \end{bmatrix} = \begin{bmatrix} c_0[m+1] \\ c_1[m+1] \end{bmatrix}$$

(d) *(5 points)* You are provided with a training sequence

$$x_n = (-1)^n$$

$n = 0, 1, 2, 3$. If $a = 1/2$, the initial tap weights are $c_0 = c_1 = 0$ and the step size is $\Delta = \frac{1}{10}$ what are the updated tap weights after you perform one iteration of the update procedure.

SOLUTION:

See equations 4.115 and 4.116 from p289 of communication system, 4th edition, for reference.

Let a[n] be the output of the channel, the convolution of the training sequence and impulse response.

From the question, $c_1[0] = c_0[0] = 0, \Delta = 1/10, a[0] = 1$

$$c_1[1] = c_1[0] + \Delta x[n-1][a[0] - \sum_{k=0}^{1} c_k[n]x[n-k]] = 0 + 0 = 0$$

$$c_0[1] = c_0[0] + \Delta x[n][a[0] - \sum_{k=0}^{1} c_k[n]x[n-k]] = 0 + 1/10 * 1 = 1/10$$