THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring**

Problem Set 8

Haykin: 3.7, 3.9-3.15

1. **Baby Quantization, Just in Case:** This is a simple problem, but it's worth doing just to make sure everyone is in synch. Suppose you have (for $t \geq 0$) a periodic signal

$$s(t) = u_{-2}(t) + 2\sum_{k=1}^{\infty}(-1)^k u_{-2}(t-2k+1)$$

for $t \geq 0$ where $u_{-2}(t)$ is the unit ramp function.

   (a) Sketch $s(t)$ over one cycle.

   SOLUTION: *refer to figure 1, where two cycles are shown.*

   (b) What might the PDF of $s(t)$ look like (use your instincts – you do NOT have the formal machinery to deal with this just yet).

   SOLUTION: *Uniform on* ± 1

   (c) $Q()$ is a one bit quantizer with $x_0 = 0$, $q_0 = -1$, $q_1 = 1$. Sketch he quantized signal $Q(s(t))$ and the error signal $s(t) - Q(s(t))$. Assume $s()$ is uniform on ± 1. Is $Q()$ an optimum one bit quantizer for $s(t)$? If not, what IS the optimum one bit quantizer for $s()$?

   SOLUTION: *Analytically we have*

$$Q(s(t)) = \sum_{k=0}^{\infty} u_{-1}(t-2k+1)$$

   for $t \geq 0$*. The Lloyd-max conditions state that* $x_0 = (q_1 + q_0)/2$ *which is true. They also state that* $q_0 = E[X|X < 0]$ *which is NOT true (should be* $-1/2$*). It's symmetric so the optimal* $q_1 = 1/2$*. So it was NOT an optimal quantizer.*

   (d) $Q()$ is a two bit quantizer with $x_k = -1/2 + k/2$, $k = 0, 1, 2$ and $q_k = -3/4 + k/2$, $k = 0, 1, 2, 3$. Sketch he quantized signal $Q(s(t))$ and the error signal $s(t) - Q(s(t))$. Is this quantizer optimal, assuming uniform $s()$ on ± 1?

   SOLUTION: *Yes, this quantizer is optimal since it satisfies the Lloyd-max optimality conditions. For sketch, refer to figure 2*

2. **Quantization Recap:** For the two bit quantizer of the previous part, code q_0 as 00, q_1 as 01, q_2 as 10 and q_3 as 11. Determine the sequence of codes which would come out of

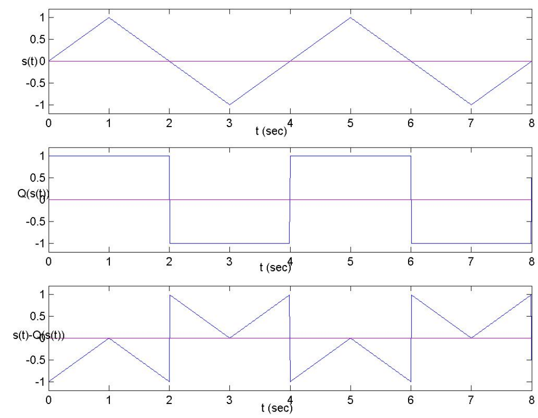

Figure 1: 1 bit quantizer, quantized signal and error signal

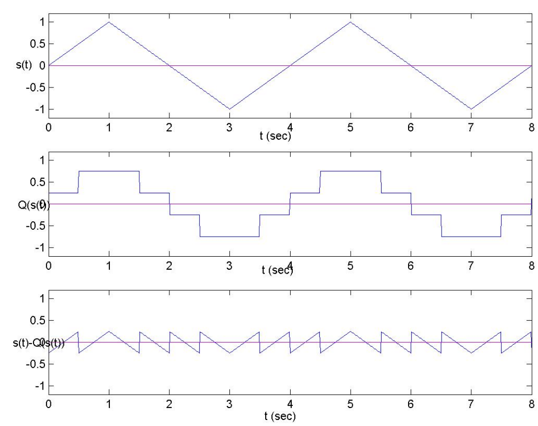

Figure 2: 2 bit quantizer

the quantizer over one cycle. Assume that samples are taken every half second and that $Q(-0.5) = -1/4$, $Q(0) = 1/4$, $Q(0.5) = 3/4$.

Discuss how one would reconstruct an approximation of the input signal using the sequence of codes at a receiver.

SOLUTION: *Sequence:* $10, 11, 11, 11, 10, 01, 00, 01, 10$. *These values would be played back through an inverse coder which maps the binary values to the quantized levels. The result would be the sketch of the quantized signal. You'd then put the signal through a perfect low pass filter (as if it were PAM). The result is a (rough) replica of the original sawtooth waveform. You might try it using Matlab. For sketch, refer to figure 3*

3. **Convexity:** Using the definition of convexity, determine for what values of α the function $f(x, y) = x^2 + \alpha xy + y^2$ is convex.

   SOLUTION: *We check for convexity by setting up the following inequality and determine if it's true.*

$$\lambda f(x_1, y_1) + (1 - \lambda) f(x_2, y_2) \geq f(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)$$

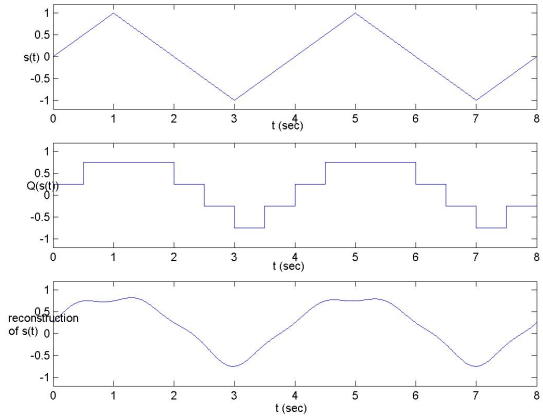

Figure 3: 2 bit quantizer and reconstruction

and examine it for all x_i and y_i. First we rewrite $f(x,y) = (x+y)^2 + (\alpha-2)xy$ which gives us

$$\begin{array}{ccc} \lambda(x_1+y_1)^2 + (1-\lambda)(x_2+y_2)^2 & & (\lambda(x_1+y_1) + (1-\lambda)(x_2+y_2))^2 \\ + & \geq & + \\ (\alpha-2)(\lambda x_1 y_1 + (1-\lambda)x_2 y_2) & & (\alpha-2)(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \end{array}$$

Exapnding and rearranging we have

$$\begin{array}{ccc} \lambda(1-\lambda)(x_1+y_1)^2 + (1-\lambda)\lambda(x_2+y_2)^2 & & 2\lambda(1-\lambda)(x_1+y_1)(x_2+y_2) \\ + & \geq & + \\ (\alpha-2)\lambda(1-\lambda)(x_1y_1 + x_2y_2) & & (\alpha-2)\lambda(1-\lambda)(x_1y_2 + x_2y_1) \end{array}$$

Rearranging again

$$[(x_1-x_2) - (y_2-y_1)]^2 \geq (\alpha-2)(x_1-x_2)(y_2-y_1)$$

which becomes

$$x^2 + y^2 \geq \alpha xy$$

with $x = (x_1 - x_2)$ and $y = (y_2 - y_1)$. Some sleight of hand: remember that $\sin\theta = x/\sqrt{x^2+y^2}$ and $\cos\theta = y/\sqrt{x^2+y^2}$ so we have

$$1 \geq \alpha \sin\theta\cos\theta = \frac{\alpha}{2}\sin 2\theta$$

Since θ can take on any value we must have $|\alpha| \leq 2$ to guarantee the inequality holds.

So $f(x,y)$ is convex if $|\alpha| \leq 2$.

4. **Delta Modulation:**

   For the sawtooth waveform

$$s(t) = u_{-2}(t) + 2\sum_{k=1}^{\infty}(-1)^k u_{-2}(t-2k+1)$$

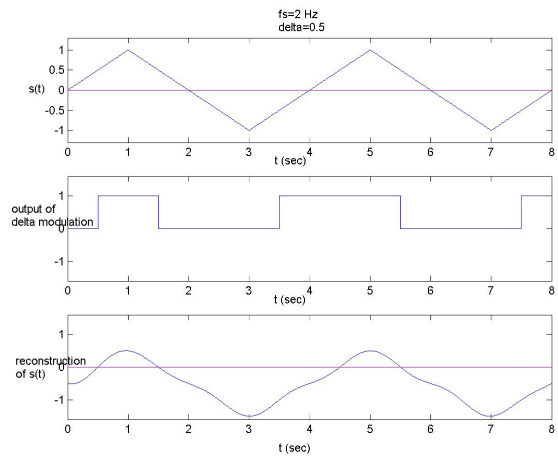

Figure 4: $f_s = 2Hz\ \Delta = 0.5$

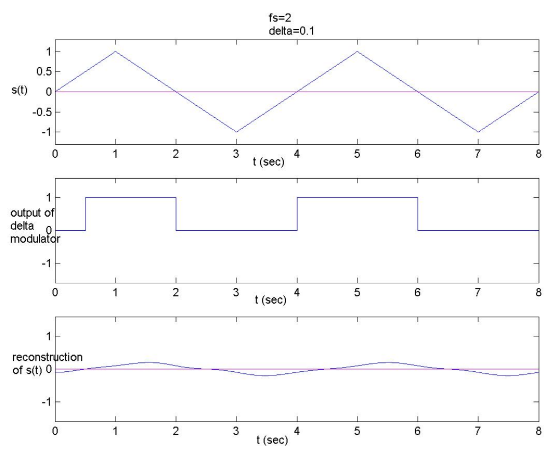

Figure 5: $f_s = 2Hz\ \Delta = 0.1$

with $t \geq 0$, make sketches of the output of a delta modulator for sample rates $2Hz$ and $10Hz$ and step sizes $\Delta = 1/2$ and $\Delta = 1/10$ (four sketches total). Also sketch the outputs of the associated demodulators.

SOLUTION:

5. **Linear Prediction:** A one-step linear predictor operates on the sampled version of a sinusoidal signal. The sampling rate is equal to $10f_o$ where f_o is the frequency of the sinusoid. The predictor has a single coefficient denoted by w_1.

   (a) Determine the optimum value of w_1 required to minimize the prediction error variance.
   SOLUTION: *Let the sinusiodal signal be $m(t) = A\sin(2\pi f_o t)$. To be able to apply the standard equations 3.66 and 3.67 of Haykin, we need to find the autocorrelation*

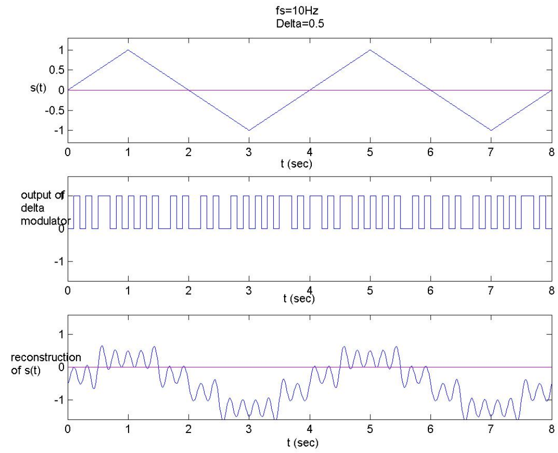

Figure 6: $f_s = 10Hz \ \Delta = 0.5$

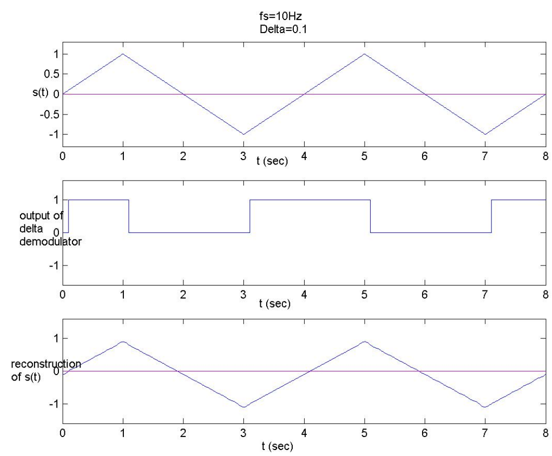

Figure 7: $f_s = 10Hz \ \Delta = 0.1$

function $R_m(\tau)$ of the above sine wave.

$$\begin{aligned}
R_m(\tau) &= E[m(t)m(t+\tau)] \\
&= E[A\sin(2\pi f_o t)A\sin(2\pi f_o(t+\tau))] \\
&= E[A^2\sin(2\pi f_o t)(\sin(2\pi f_o t)\cos(2\pi f_o \tau) - \sin(2\pi f_o \tau)\cos(2\pi f_o t)) \\
&= A^2 E[(1-\cos(4\pi f_o t))/2\cos(2\pi f_o \tau) - 1/2\sin(2\pi f_o \tau)(\sin(4\pi f_o t) - \sin(0))] \\
&= \frac{A^2}{2}\cos(2\pi f_o \tau)
\end{aligned}$$

where the last step follows from the fact that the signal is a deterministic one and hence the expected value is over one period.

Hence,

$$R_m(0) = \frac{A^2}{2}$$

$$R_m(1) = \frac{A^2}{2}\cos(2\pi f_o \frac{1}{2\pi 10 f_o}) = \frac{A^2}{2}\cos(0.1)$$

Thus for this problem we have,

$$R_m = [R_m(0)], r_m = [R_m(1)]$$

Hence, the optimum solution is given by,

$$\begin{aligned} w_o &= R_m^{-1} r_m \\ &= cos(0.1) \\ &= 0.995 \end{aligned}$$

(b) Determine the minimum value of prediction error variance.

SOLUTION:

$$\begin{aligned} J_{min} &= R_m(0) - r_m^T R_m^{-1} r_m \\ &= \frac{A^2}{2} - \frac{\frac{A^2}{2}\cos(0.1)\frac{A^2}{2}\cos(0.1)}{((A^2)/2)} \\ &= \frac{A^2}{2}(1 - \cos^2(0.1)) \\ &= 0.005A^2 \end{aligned}$$

HINT: This is a direct application of linear prediction filter we studied in section 3.13. To be able to apply the standard equation, we need to find the autocorrelation funtion of the sinusiod of the form $m(t) = A\sin(2\pi f_o t)$. Note that the sinusiod being deterministic, the expectation is over one time period.